

Connectedness of geometric representation of substitutions of Pisot type

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Abstract

We give some criterions to establish connectedness for the geometric representation domain of substitutions of Pisot type, unimodular and satisfying the strong coincidence condition (PUC substitutions).

1 Introduction

Every substitution of Pisot type, unimodular and satisfying the strong coincidence condition admits a geometric representation as a subset \mathcal{F} of the euclidian space $\mathbb{R}^{r-1} \times \mathbb{C}^s$ with a piecewise translation \mathcal{T} . The structure and the properties of these geometric representations have been intensively studied [AI01, HZ98, SW99, CS01b] since G. Rauzy gave the first example on 3 letters [Rau82] (Sturmian substitutions are very well known examples on 2 letters [MH38]). Indeed, G. Rauzy proved that the representation domain \mathcal{F} associated to the substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ is a connected and simply connected subset of \mathbb{C} .

Theorem 3.1 gives a necessary and sufficient condition for the representation domain to be connected. However, this criterion states that an infinite number of conditions have to be satisfied to obtain connectedness. Hence this result is more useful to prove the non-connectedness of some representations (see Example 2), as we only have to show that one condition is not satisfied.

Inspired by G. Rauzy's proof for connectedness of \mathcal{F} , we give a second sufficient criterion of connectedness, Theorem 3.2. This result permits to conclude even if the

Received by the editors December 2001.

Communicated by H. Van Maldeghem.

2000 *Mathematics Subject Classification* : Primary 37B10, 28A80.

Key words and phrases : substitutions, geometric representation, Pisot type, connectedness.

image \mathcal{F}_a of a cylinder $[a]$ is the union of a finite number of connected components, as it is the case for $1 \mapsto 12312$, $2 \mapsto 132$, $3 \mapsto 2$ (see Figure 4). Moreover, that criterion can be explicitly verified by computations on a graph given by A. Siegel, and the representation connectedness thus proved by the existence of a given subset of edges in a finite graph [Sie].

2 Generalities

2.1 Generalities and notations

Let $\mathcal{A} = \{1, 2, \dots, d\}$ ($d \geq 2$) a finite alphabet, \mathcal{A}^* the set of finite words on \mathcal{A} . We denote the set of non-empty finite words $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$, where ε design the empty word of \mathcal{A}^* . We call *bi-infinite word* every element w in $\mathcal{A}^{\mathbb{Z}}$. We write such a word by pointing it between w_{-1} and w_0 , as following $w = \dots w_{-2}w_{-1}.w_0w_1\dots$. We denote $|U|$ the length of the word U and for all letter $a \in \mathcal{A}$, we denote $|U|_a$ the occurrences number of letter a in the word U . The map $l : \mathcal{A}^* \rightarrow \mathbb{N}^d$ which to a word U associate the vector $(|U|_i)_{i=1,\dots,d}$ is called *Parikh map* and its values are the *Parikh vectors*.

A *substitution* is a morphism σ for the concatenation operation of the free monoid \mathcal{A}^* , which maps \mathcal{A} on \mathcal{A}^+ and such that there exists a letter a in \mathcal{A} satisfying $\lim_{n \rightarrow \infty} |\sigma^n(a)| = +\infty$. The substitution σ can be naturally prolonged to the set of infinite words $\mathcal{A}^{\mathbb{Z}}$ by

$$\sigma(\dots w_{-2}w_{-1}.w_0w_1\dots) = \dots \sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\dots$$

We denote S the *shift* on $\mathcal{A}^{\mathbb{Z}}$, which to every word $w = (w_i)_{i \in \mathbb{Z}}$ maps the words $Sw = (w_{i+1})_{i \in \mathbb{Z}}$. We call *S-periodic point* every word w in $\mathcal{A}^{\mathbb{Z}}$ such that there exists $h \geq 1$ with $S^h(w) = w$. A substitution σ is said *S-periodic* if there exists a periodic point of σ which is also *S-periodic*.

A substitution σ is said *primitive* if there exists a natural integer k such that b occurs in $\sigma^k(a)$ for all couple (a, b) in \mathcal{A}^2 .

To every substitution σ we canonically associate the *occurrence matrix* with non-negative integer coefficients $M_\sigma = (m_{i,j})_{1 \leq i,j \leq d}$ defined by

$$m_{i,j} = |\sigma(j)|_i,$$

and P_σ its characteristic polynomial. The maps σ and M_σ satisfy

$$\forall w \in \mathcal{A}^* : \quad l(\sigma(w)) = M_\sigma l(w). \quad (1)$$

An algebraic integer is called *Pisot-Vijayaraghavan number* or *Pisot number* if all its algebraic conjugates β are such that $|\beta| < 1$. A substitution σ is said of *Pisot type* if its characteristic polynomial P_σ is irreducible over \mathbb{Q} and its dominating eigenvalue is a Pisot number. Moreover, σ is said *unimodular* if $\det M_\sigma = \pm 1$.

All substitutions of Pisot type are primitive and non-*S-periodic*.

2.2 Dynamical symbolical system and coding

One can report to [CS01a] and [CS01b] for more details.

Definition 2.1. [CS01a] Let Γ be the map $\Gamma : \Omega \rightarrow (\mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)^{\mathbb{N}}$ which to all $w \in \Omega$ associate the sequence $(p_i, a_i, s_i)_{i \geq 0}$ (with $\sigma(a_{i+1}) = p_i a_i s_i$ for all $i \geq 0$). $\Gamma(w) = (p_i, a_i, s_i)_{i \geq 0}$ is called **prefix-suffix development of w** .

The different developments are the labels of the paths in the following automaton (see [CS01a]).

Definition 2.2. [CS01a] The **prefix-suffix automaton A_σ** associated to the substitution σ is such that

- \mathcal{A} is the state set; all states are initial,
- $\mathcal{P} = \{(p, a, s); \exists b \in \mathcal{A} \text{ such that } \sigma(b) = pas\}$ is the label set,
- there exists an edge from a to b labelled $e = (p, a, s)$ if $pas = \sigma(b)$, that is $a \xrightarrow{(p,a,s)} b$.

We call \mathcal{D} the set of labels of infinite walks in this automaton.

We have the following theorem

Theorem 2.1. [CS01a] Let σ be a primitive substitution without S -periodic fixed point. Let (Ω, S) ($\Omega \subset \mathcal{A}^{\mathbb{Z}}$) be the dynamical system associated to σ . The map Γ defined above is continuous and onto on the subshift of finite type \mathcal{D}_σ . It is one-to-one except maybe on a countable set of points, more precisely except on the S -orbit of periodic points for σ .

Let σ be a unimodular substitution of Pisot type. Let α_1 the Pisot eigenvalue, $\{\alpha_2, \dots, \alpha_r\}$ the real contracting eigenvalues, and $\{\alpha_{r+1}, \dots, \alpha_{r+s}, \overline{\alpha_{r+1}}, \dots, \overline{\alpha_{r+s}}\}$ the complex contracting eigenvalues.

We recall (see [CS01b]) that the **digit** map δ is a morphism for the concatenation operation of \mathcal{A}^* :

$$\delta(w_1 w_2) = \delta(w_1) + \delta(w_2).$$

Moreover, if $\mathbf{C} \in \mathcal{M}_{s+r-1}(\mathbb{C})$ is the diagonal matrix of size $s + r - 1$ whose entries are the contracting eigenvalues (conjugate complex eigenvalues are taken only once): we have $\|\mathbf{C}\| < 1$. Then \mathbf{C} and σ verify a commutation relation:

$$\delta(\sigma(j)) = \mathbf{C} \delta(j) \quad \text{with } \mathbf{C} = \begin{pmatrix} \alpha_2 & & (0) \\ & \ddots & \\ (0) & & \alpha_{r+s} \end{pmatrix}.$$

Definition 2.3. [CS01b] Let $c = (p_j, a_j, s_j)_{j \geq 0}$ in \mathcal{D} a walk in the prefix-suffix automaton. Let Λ the continuous map defined from \mathcal{D} to $\mathbb{R}^{r-1} \times \mathbb{C}^s$ by

$$\begin{aligned} \Lambda(c) &= \lim_{n \rightarrow +\infty} \delta(\sigma^n(p_n) \dots \sigma^0(p_0)) \\ &= \sum_{j \geq 0} \mathbf{C}^j \delta(p_j) = \begin{pmatrix} \sum_{j \geq 0} \delta_2(p_j) \alpha_2^j \\ \vdots \\ \sum_{j \geq 0} \delta_{r+s}(p_j) \alpha_{r+s}^j \end{pmatrix}. \end{aligned}$$

Definition 2.4. [CS01b] Let x be a word of Ω and $\Gamma(x) = (p_j, a_j, s_j)_{j \geq 0}$ its prefix-suffix development. Let φ be the continuous **representation map**, defined from Ω to $\mathbb{R}^{r-1} \times \mathbb{C}^s$:

$$\begin{aligned} \varphi(x) &= \Lambda \circ \Gamma(x) \\ &= \sum_{j \geq 0} \mathbf{C}^j \delta(p_j). \end{aligned}$$

Proposition 2.1. [CS01b] For all x in Ω :

$$\begin{aligned} \varphi(Sx) &= \varphi(x) + \delta(x_0) \\ \varphi(\sigma x) &= \mathbf{C} \varphi(x). \end{aligned}$$

We recall the strong coincidence condition:

Definition 2.5. [AI01] A substitution σ satisfies the strong coincidence condition on the prefixes (resp. on the suffixes) if for all couple of letter (j, k) there exists a constant n such that $\sigma^n(j)$ and $\sigma^n(k)$ can be decomposed in the following way

$$\begin{aligned} \sigma^n(j) &= pas \text{ and } \sigma^n(k) = qar, \text{ with } l(p) = l(q) \\ &\text{(resp. } l(s) = l(r)) \end{aligned}$$

We denote PUC a substitution of Pisot type, unimodular and satisfying the strong coincidence condition. For a PUC substitution (see [CS01b]), the representation map φ is one-to-one in measure, which implies that the image of Ω by φ can be decomposed as a disjoint union

$$\mathcal{F} = \coprod_{i=1}^d \varphi[i] \quad \text{almost everywhere.}$$

If we denote $\mathcal{F}_i = \varphi[i]$ the image of the cylinder $[i]$, we can define almost everywhere a piecewise exchange \mathcal{T} on \mathcal{F} ,

$$\begin{aligned} \mathcal{T} : \quad \mathcal{F} &\rightarrow \mathcal{F} \\ x \in \mathcal{F}_i &\mapsto x + \delta(i). \end{aligned} \tag{2}$$

We then have a measure-theoretically isomorphism between (Ω, S) and piecewise translation on a self-similar domain.

Theorem 2.2. [AI01, CS01b] Let σ be a PUC substitution, then the symbolical dynamical system $(\Omega, S, \mathcal{B}_\Omega, \mu)$ is measure theoretically isomorphic to the system $(\mathcal{F}, \mathcal{T}, \mathcal{B}_{\mathbb{R}^{r-1} \times \mathbb{C}^s}, m)$.

The shift S on Ω is then conjugate in measure to \mathcal{T} , whereas the substitution σ is conjugate to the contraction \mathbf{C} .

Moreover, the domain \mathcal{F} has a self-similar structure. Let p be a word over \mathcal{A} , we call f_p the contraction

$$\begin{aligned} f_p : \mathbb{R}^{r-1} \times \mathbb{C}^s &\rightarrow \mathbb{R}^{r-1} \times \mathbb{C}^s \\ x &\mapsto \mathbf{C}x + \delta(p). \end{aligned} \tag{3}$$

The description of cylinders of Ω as $[i] = \prod_{i \xrightarrow{(p,i,s)} j} S^{|p|} \sigma[j]$ implies that the images of cylinders can be decomposed in measure in the following way

$$\mathcal{F}_i = \prod_{i \xrightarrow{(p,i,s)} j} f_p(\mathcal{F}_j), \quad (4)$$

where \mathcal{F}_i are compact subsets.

3 Connectedness of the representation domain \mathcal{F}

In the following, except when explicitly mentioned, we will consider **PUC substitutions over the alphabet \mathcal{A}** . We recall that A_σ is the prefix-suffix automaton associated to σ .

Example 1 Let σ be the PUC substitution $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$. Let β be a contracting eigenvalue of M_σ , that is a complex root of the polynomial $x^3 = x^2 + x + 1$. The matrix \mathbf{C} is (β) and the vector $\delta = (\delta(1), \delta(2), \delta(3))$ can be chosen as $(1, \beta - 1, \beta^2 - \beta - 1)$. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ (see Figure 3) has the following self-similar structure

$$\begin{aligned} \mathcal{F}_1 &= \beta \mathcal{F}_1 \cup (\beta \mathcal{F}_2 + \beta^2 - \beta - 1) \cup \beta \mathcal{F}_3 \\ \mathcal{F}_2 &= \beta \mathcal{F}_1 + 1 \\ \mathcal{F}_3 &= \beta \mathcal{F}_2 \end{aligned} \quad (5)$$

■

We define some particular sets named bricks.

Definition 3.1. Let $e_1 \dots e_k$ be a path of length k in the automaton A_σ , from a to b , with $e_i = (p_i, a_i, s_i)$. We call **k -brick** associated to this path the set

$$\begin{aligned} B(a) &= \mathcal{F}_a && \text{if } k = 0, \\ B(a; e_1 \dots e_k; b) &= f_{p_1} \circ \dots \circ f_{p_k}(\mathcal{F}_b) && \text{if } k \geq 1. \end{aligned} \quad (6)$$

Example 1 (continuation) The system (5) can be written in the following way with bricks (see Figure 1)

$$\begin{aligned} B(1) &= B(1; (\varepsilon, 1, 2); 1) \cup B(1; (3, 1, \varepsilon); 2) \cup B(1; (\varepsilon, 1, \varepsilon); 3), \\ B(2) &= B(2; (1, 2, \varepsilon); 1), \\ B(3) &= B(3; (\varepsilon, 3, 1); 2). \end{aligned}$$

■

The bricks have the following properties. Denote $D = \text{diam}(\mathcal{F})$. It follows obviously from (4) that

$$B(a; e_1 \dots e_k; b) \subset \mathcal{F}_a, \quad (7)$$

$$B(a; e_1 \dots e_k; b) = \bigcup_{b \xrightarrow{e} c} B(a; e_1 \dots e_k e; c), \quad (8)$$

$$\text{diam}(B(a; e_1 \dots e_k; b)) \leq \|\mathbf{C}\|^k D. \quad (9)$$

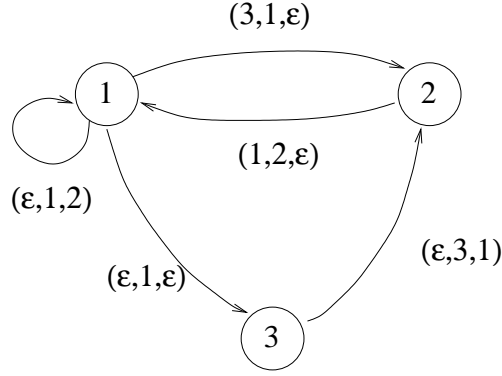


Figure 1: Prefix-suffix automaton of $\sigma : 1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$

We can easily remark that, for any path $P \in \mathcal{D}$ with $P = (e_i)_{i \geq 1}$ and $e_i = (p_i, a_i, s_i)$, we have

$$\Lambda(P) \in B(a_1; e_1 \dots e_i; a_{i+1}) \quad \forall i \geq 1. \quad (10)$$

Similarly, let the path $P' = (e'_i)_{i \geq 1}$ where $e'_1 = (p, a, s)$ and for all $i \geq 2$, $e'_i = e_{i-1}$. Then

$$\Lambda(P') = f_p(\Lambda(P)) = \delta(p) + \mathbf{C}\Lambda(P). \quad (11)$$

In particular, for any path $P \in \mathcal{D}$, it is possible to give a path P' with $p = \varepsilon$. That means that for all $z \in B(a_1)$, we have $\mathbf{C}z \in \mathbf{C}B(a_1) = B(a; (\varepsilon, a, s); a_1)$.

For any integer $k \geq 0$, denote \mathcal{B}_k the set of k -bricks associated to paths of length k . Hence $\mathcal{B}_0 = \{\mathcal{F}_a; a \in \mathcal{A}\}$ and

$$\begin{aligned} \mathcal{B}_1 &= \{B(a; (p, a, s); b) \ ; \ \sigma(b) = pas\} \\ &= \{f_p(\mathcal{F}_b) \ ; \ (p, a, s) \in \mathcal{P}, \sigma(b) = pas\}. \end{aligned}$$

In order to define an equivalence relation on the sets \mathcal{B}_k , we define a first relation \mathcal{R}' .

Definition 3.2. *Two subsets E_1 and E_2 of $\mathbb{R}^{r-1} \times \mathbb{C}^s$ are in relation by \mathcal{R}' and noted $E_1 \mathcal{R}' E_2$ if*

$$E_1 \cap E_2 \neq \emptyset.$$

The relation \mathcal{R}' is trivially reflexive and symmetrical, so it is possible for any positive integer k to define the following equivalence relations by completing \mathcal{R}' by transitivity

- let $\mathcal{R}^{(k)}$ the equivalence relation defined by completing \mathcal{R}' by transitivity on the set of k -bricks,

- for any letter a in \mathcal{A} , let $\mathcal{R}_a^{(k)}$ the equivalence relation defined by completing \mathcal{R}' by transitivity on the set of k -bricks of the form $B(a; e_1 \dots e_k; b)$.

We give a necessary and sufficient criterion for the connectedness of \mathcal{F} .

Theorem 3.1. *Let σ be a PUC substitution and $(\mathcal{F}, \mathcal{T})$ the geometric representation generated by σ . Then \mathcal{F} is connected if and only if the following condition is satisfied*

$$\forall k \geq 0 : \quad \forall B_1, B_2 \in \mathcal{B}_k : \quad B_1 \mathcal{R}^{(k)} B_2. \quad (12)$$

Proof. We will show that the set \mathcal{F} is not connected if and only if there exists an integer $k \geq 0$ such that all bricks in \mathcal{B}_k are not in the same equivalence class for the relation $\mathcal{R}^{(k)}$.

Suppose there exists $k \geq 0$ and $B_1, B_2 \in \mathcal{B}_k$ such that $B_1 \not\mathcal{R}^{(k)} B_2$, then the set of bricks \mathcal{B}_k can be divided in at least two equivalence classes for $\mathcal{R}^{(k)}$. Let $n \geq 2$ the number of classes, which is finite as $\#\mathcal{B}_k$ is finite. Denote C_i with $1 \leq i \leq n$ the union of the elements in each class. Then the sets C_i and C_j are disjoint one to another. The sets C_i are closed, as finite unions of compact subsets. It suffices then to remember that $\mathcal{F} = \bigcup_{1 \leq i \leq n} C_i$ to see that \mathcal{F} is not connected.

For the converse, suppose that \mathcal{F} is not connected. It is compact, so the union of at least two compact connected components. Let's consider for simplicity that the number of connected components is two, which we note C_1 and C_2 . We have to show that C_1 and C_2 are unions of k -bricks for a given k . Let Δ be the euclidian distance on $\mathbb{R}^{r-1} \times \mathbb{C}^s$. The compactness of C_1 and C_2 says that $\Delta(C_1, C_2) = g > 0$. Then, for all $x_1 \in C_1$ and $x_2 \in C_2$, $\Delta(x_1, x_2) \geq g$. As $\|\mathbf{C}\| < 1$, there exists an integer k such that $\|\mathbf{C}\|^k D < g$. Let now B be a k -brick, and y_1, y_2 be two points from B . The relation (9) proves that $\Delta(y_1, y_2) \leq \|\mathbf{C}\|^k D < g$, which implies that $B \subset C_1$ or $B \subset C_2$. Hence, the sets C_1 and C_2 are unions of k -bricks. Finally, no brick $B_1 \subset C_1$ is in relation by \mathcal{R}' with a brick $B_2 \subset C_2$, as $g > 0$, and so by $\mathcal{R}^{(k)}$. ■

Example 2 Let σ be the PUC substitution $1 \mapsto 112, 2 \mapsto 21$. The representation domain \mathcal{F} can be decomposed in the following way

$$\begin{aligned} B(1) &= B(1; (\varepsilon, 1, 12); 1) \cup B(1; (1, 1, 2); 1) \cup B(1; (2, 1, \varepsilon); 2) \\ B(2) &= B(2; (\varepsilon, 2, 1); 2) \cup B(2; (11, 2, \varepsilon); 1) \end{aligned}$$

We can verify by computation that for $k = 2$, the condition (12) is not satisfied, as for example the 2-brick $B(1; (2, 1, \varepsilon)(\varepsilon, 2, 1); 2)$ is not in relation with any other 2-brick. This shows that the geometric representation domain \mathcal{F} is not connected.

Figure 2 shows a drawing of the geometric representation of the substitution $\sigma : 1 \mapsto 112, 2 \mapsto 21$, where \mathcal{F}_1 is shown in clear and \mathcal{F}_2 in dark. The subsets are not on the same level to improve visibility. The two vertical barres show a mesh of the lattice $\mathcal{L} = \{k(\delta(2) - \delta(1)) \mid k \in \mathbb{Z}\}$. Despite what may appear on the figure, $m(\mathcal{F}_1 \cap \mathcal{F}_2) = 0$. ■



Figure 2: Geometric representation of the substitution $\sigma : 1 \mapsto 112, 2 \mapsto 21$

We formulate a sufficient condition for \mathcal{F} to be connected.

Theorem 3.2. *Let σ be a PUC substitution and $(\mathcal{F}, \mathcal{T})$ the geometric representation generated by σ . If the two following conditions are satisfied*

$$\forall a_1, a_2 \in \mathcal{A} : \quad B(a_1)\mathcal{R}^{(0)}B(a_2), \quad (13)$$

$$\forall a \in \mathcal{A} : \forall (e, f) \in \mathcal{P}^2 \text{ couple of edges starting from } a : \quad (14)$$

$$B(a; e; b)\mathcal{R}_a^{(1)}B(a; f; c),$$

then \mathcal{F} is connected. Moreover, the domain \mathcal{F}_a is connected for all $a \in \mathcal{A}$.

Remark that condition (13) is also necessary for the connectedness of \mathcal{F} , but we will see later that if \mathcal{F} is connected but not all \mathcal{F}_a , then condition (14) is not satisfied.

Proof. The proof is based on G. Rauzy ([Rau82]) and A. Messaoudi ([Mes96]) works on the substitution $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$. As \mathcal{F} is a compact set, it suffices to show that it is well chained. For this we construct, for all $\epsilon > 0$ and any couple of points (x, y) in \mathcal{F}^2 , a chain of points x_1, \dots, x_n in \mathcal{F} , linking x to y ($x_1 = x$ and $x_n = y$), and such that for all $1 \leq j \leq n - 1$, we have $d(x_j, x_{j+1}) < \epsilon$.

We treat in a first step the case of x and y being in the same subset \mathcal{F}_a . We show by induction on h that for any letter a in \mathcal{A} and any couple (x, y) in \mathcal{F}_a^2 , there exists a chain x_1, \dots, x_n linking x to y ($x_1 = x$ and $x_n = y$), such that for all $1 \leq j \leq n - 1$, there exists a path $e_1 \dots e_h$ from a to b_j of length h in A_σ such that $x_j, x_{j+1} \in B(a; e_1 \dots e_h; b_j)$. Finally, with relation (9) it suffices to take h big enough such that $\|\mathbf{C}\|^h D < \epsilon$.

Let $a \in \mathcal{A}$ and $(x, y) \in \mathcal{F}_a^2$. The case $h = 0$ is obvious by relation (6) and x, y is a chain linking x to y .

Assume that the property is true for some $h \geq 0$. By hypothesis, there exists a chain x_1, \dots, x_n linking x to y ($x_1 = x$ and $x_n = y$), such that for all $1 \leq j \leq n - 1$, there exists a path $e_1 \dots e_h$ from a to b of length h such that $x_j, x_{j+1} \in B(a; e_1 \dots e_h; b)$. Note here that $e_1 \dots e_h$ and b depend on j . Let $j \in \{1, \dots, n - 1\}$ be fixed. From (8), we have $B(a; e_1 \dots e_h; b) = \bigcup_{b \xrightarrow{e} c} B(a; e_1 \dots e_h e; c)$. Two cases arise:

1st case: there exists an edge $e \in \mathcal{P}$ such that $x_j, x_{j+1} \in B(a; e_1 \dots e_h e; c)$, then $e_1 \dots e_h e$ is a path from a to c of length $h + 1$, such that $x_j, x_{j+1} \in B(a; e_1 \dots e_h e; c)$.

2nd case: such an edge does not exist. Then there exist two edges e^j, e^{j+1} from b to c^j and c^{j+1} such that $x_j \in B(a; e_1 \dots e_h e^j; c^j)$ and $x_{j+1} \in B(a; e_1 \dots e_h e^{j+1}; c^{j+1})$.

Let

$$\begin{aligned} x' &= f_{p_h}^{-1} \circ \cdots \circ f_{p_1}^{-1}(x_j) \in B(b; e^j; c^j) & \text{and} \\ y' &= f_{p_h}^{-1} \circ \cdots \circ f_{p_1}^{-1}(x_{j+1}) \in B(b; e^{j+1}; c^{j+1}). \end{aligned}$$

From (14) we know that

$$B(b; e^j; c^j) \mathcal{R}_b^{(1)} B(b; e^{j+1}; c^{j+1}),$$

this implies that there exists a chain y_1, \dots, y_m of length m linking x' to y' ($y_1 = x'$ and $y_m = y'$) such that for all $1 \leq i \leq m-1$, there exists an edge $g^i = (q^i, b, s^i)$ ($g^1 = e^j$ and $g^{m-1} = e^{j+1}$) from b to d^i , with $y_i, y_{i+1} \in B(b; g^i; d^i)$. This implies that

$$\left. \begin{array}{l} f_{p_1} \circ \cdots \circ f_{p_h}(y_i) \\ f_{p_1} \circ \cdots \circ f_{p_h}(y_{i+1}) \end{array} \right\} \in B(a; e_1 \dots e_h g^i; d^i).$$

Between x_j and x_{j+1} we insert the chain $(f_{p_1} \circ \cdots \circ f_{p_h}(y_2), \dots, (f_{p_1} \circ \cdots \circ f_{p_h}(y_{m-1}))$. Thus the property is satisfied for $h+1$.

Let now $(x, y) \in \mathcal{F}^2$. By (13), there exists a chain z_1, \dots, z_l ($z_1 = x$ and $z_l = y$) of length l , such that for all $1 \leq i \leq l-1$, we have $z_i, z_{i+1} \in B(a_i) = \mathcal{F}_{a_i}$. Since the property is verified for any couple $(z_i, z_{i+1}) \in \mathcal{F}_{a_i}^2$, it is satisfied also for any couple $(x, y) \in \mathcal{F}^2$, and this ends the proof. \blacksquare

Example 1 (continuation) We give explicitly the intersection points of the bricks.

We give the automaton paths for the substitution $1 \mapsto 12$, $2 \mapsto 31$, $3 \mapsto 1$, which allow to verify that the geometric representation domain \mathcal{F} is connected.

To show for example that $B(1) \mathcal{R}^{(0)} B(2)$, let

$$\begin{aligned} c_1 &= (\varepsilon, 1, 2)(\varepsilon, 1, \varepsilon)(\varepsilon, 3, 1)[(1, 2, \varepsilon)(3, 1, \varepsilon)]^\infty, \\ c_2 &= (1, 2, \varepsilon)(\varepsilon, 1, 2)(\varepsilon, 1, 2)(3, 1, \varepsilon)[(1, 2, \varepsilon)(3, 1, \varepsilon)]^\infty. \end{aligned}$$

Then

$$\begin{aligned} \Lambda(c_1) &= \beta^3 \left(\sum_{i \geq 0} (\delta(1) + \beta \delta(3)) \beta^{2i} \right) \\ &= \beta^3 (1 + \beta^3 - \beta^2 - \beta) \frac{1}{1 - \beta^2} \\ &= -\beta - \beta^2 \end{aligned}$$

and

$$\begin{aligned} \Lambda(c_2) &= \delta(1) + \beta^3 \delta(3) + \beta^4 \left(\sum_{i \geq 0} (\delta(1) + \beta \delta(3)) \beta^{2i} \right) \\ &= 1 + (\beta^5 - \beta^4 - \beta^3) + \beta(-\beta - \beta^2) \\ &= 1 - \beta^3 \\ &= -\beta - \beta^2 \end{aligned}$$

so $\Lambda(c_1) = \Lambda(c_2) \in B(1) \cap B(2)$. Next

$$\begin{aligned} B(1)\mathcal{R}^{(0)}B(2) & \text{ since } -\beta - \beta^2 \in B(1) \cap B(2), \\ B(1)\mathcal{R}^{(0)}B(3) & \text{ since } -\beta^2 \in B(1) \cap B(3). \end{aligned}$$

Thus (13) is satisfied. Moreover

$$\begin{aligned} B(1; (\varepsilon, 1, 2); 1) & \mathcal{R}_1^{(1)} B(1; (\varepsilon, 1, \varepsilon); 3) \\ & \text{ since } -\beta^3 \in B(1; (\varepsilon, 1, 2); 1) \cap B(1; (\varepsilon, 1, \varepsilon); 3), \\ B(1; (\varepsilon, 1, \varepsilon); 3) & \mathcal{R}_1^{(1)} B(1; (3, 1, \varepsilon); 2) \\ & \text{ since } \beta - 1 \in B(1; (\varepsilon, 1, \varepsilon); 3) \cap B(1; (3, 1, \varepsilon); 2). \end{aligned}$$

Relation (14) is verified so \mathcal{F} is connected, and moreover \mathcal{F}_1 , \mathcal{F}_2 and \mathcal{F}_3 are connected. \blacksquare

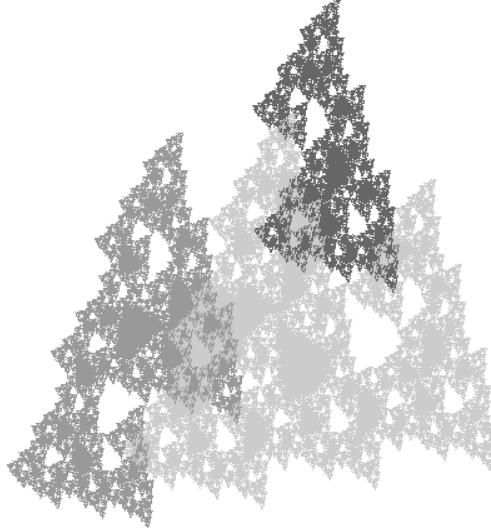


Figure 3: Representation domain of $\sigma : 1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$

The Theorem 3.2 criterion is sufficient but not necessary. Indeed, we give a second example of substitution which representation domain is connected, but which does not verify condition (14).

Example 3 Let σ be the substitution $1 \mapsto 12312, 2 \mapsto 132, 3 \mapsto 2$. The set \mathcal{F} (see Figure 4) admits the following decomposition

$$\begin{aligned} B(1) &= B(1; (\varepsilon, 1, 2312); 1) \cup B(1; (123, 1, 2); 1) \cup B(1; (\varepsilon, 1, 32); 2) \\ B(2) &= B(2; (1, 2, 312); 1) \cup B(2; (1231, 2, \varepsilon); 1) \cup B(2; (13, 2, \varepsilon); 2) \\ & \quad \cup B(2; (\varepsilon, 2, \varepsilon); 3) \cup B(2; (\varepsilon, 2, \varepsilon); 3) \\ B(3) &= B(3; (12, 3, 12); 1) \cup B(3; (1, 3, 2); 2). \end{aligned}$$

This system verifies (12) but not (14). Indeed $B(3; (12, 3, 12); 1)$ and $B(3; (1, 3, 2); 2)$ possess no intersection point, and since these are the two only bricks in $B(3)$, clearly $B(3; (12, 3, 12); 1) \mathcal{R}_3^{(1)} B(3; (1, 3, 2); 2)$. However we show in the following that \mathcal{F} is connected. ■

We give a second criterion of connectedness.

Theorem 3.3. *Let σ be a substitution on an alphabet $\mathcal{A} = \{1, \dots, d\}$. If there exists an alphabet $\overline{\mathcal{A}} = \{1_1, \dots, 1_{n_1}, \dots, d_1, \dots, d_{n_d}\}$, a substitution $\overline{\sigma}$ defined on $\overline{\mathcal{A}}$ and a letter-to-letter projection π from $\overline{\mathcal{A}}$ onto \mathcal{A} such that for all i in \mathcal{A} and all $1 \leq k \leq n_i$:*

$$\begin{aligned}\pi(i_k) &= i, \\ \pi(\overline{\sigma}(i_k)) &= \sigma(i),\end{aligned}$$

and that it is possible to express the self-similar system (4) of \mathcal{F} with

$$\mathcal{F}_i = \bigcup_{1 \leq k \leq n_i} \mathcal{F}_{i_k},$$

in such a way that

$$\mathcal{F}_{i_k} = \prod_{i_k \xrightarrow{(p, i_k, s)} j_h} f_p(\mathcal{F}_{j_h}), \quad (15)$$

and that the new system (15) verifies conditions (13) and (14), then \mathcal{F} is connected and moreover \mathcal{F}_i is the union of at most n_i connected components.

If moreover $\overline{\mathcal{A}}$ is of minimal size, then \mathcal{F}_i is the union of exactly n_i connected components.

Proof. Suppose the existence of $\overline{\mathcal{A}}$, $\overline{\sigma}$ and π such that

$$\mathcal{F}_i = \bigcup_{1 \leq k \leq n_i} \mathcal{F}_{i_k}.$$

To be coherent, we have to extend the definition of δ to the new alphabet $\overline{\mathcal{A}}$, so we pose $\delta(i_k) = \delta(i)$. Then

$$\mathcal{F}_{i_k} = \prod_{i_k \xrightarrow{(p, i_k, s)} j_h} f_p(\mathcal{F}_{j_h}).$$

By hypothesis, this system verifies the conditions (13) and (14), hence by Theorem 3.2 the set

$$\mathcal{F} = \bigcup_{1 \leq i \leq d} \mathcal{F}_i = \bigcup_{1 \leq i \leq d} \bigcup_{1 \leq k \leq n_i} \mathcal{F}_{i_k}$$

is connected. The subsets \mathcal{F}_{i_k} are also connected, which implies that for any letter i in \mathcal{A} , $\mathcal{F}_i = \bigcup_{1 \leq k \leq n_i} \mathcal{F}_{i_k}$ is the union of at most n_i connected components.

Moreover if we minimize the size of $\overline{\mathcal{A}}$, that is to say that for all couple (i_k, i_h) in $\overline{\mathcal{A}}^2$ we have $\mathcal{F}_{i_k} \cap \mathcal{F}_{i_h} = \emptyset$, then the following union is disjoint $\mathcal{F}_i = \prod_{1 \leq k \leq n_i} \mathcal{F}_{i_k}$ and \mathcal{F}_i is the union of exactly n_i connected components. ■

Example 3 (continuation) The bricks $B(3; (12, 3, 12); 1)$ and $B(3; (1, 3, 2); 2)$ are in two different equivalence classes for $\mathcal{R}_3^{(1)}$. One has to "double" the letter 3. It suffices to take the alphabet $\overline{\mathcal{A}} = \{1, 2, 3_a, 3_b\}$. This can be seen as studying a new substitution $\overline{\sigma}$ given by

$$\begin{aligned} 1 &\mapsto 123_a12 \\ 2 &\mapsto 13_b2 \\ 3_a &\mapsto 2 \\ 3_b &\mapsto 2 \end{aligned}$$

And to describe the system associated to this substitution. The substitution $\overline{\sigma}$ is degenerate in the way that $sp(M_{\overline{\sigma}}) = sp(M_{\sigma}) \cup \{0\}$, hence $\overline{\sigma}$ is not of Pisot type. Nevertheless the system (15) can be described by

$$\begin{aligned} B(1) &= B(1; (\varepsilon, 1, 23_a12); 1) \cup B(1; (123_a, 1, 2); 1) \\ &\quad \cup B(1; (\varepsilon, 1, 3_b2); 2) \\ B(2) &= B(2; (1, 2, 3_a12); 1) \cup B(2; (123_a1, 2, \varepsilon); 1) \\ &\quad \cup B(2; (13_b, 2, \varepsilon); 2) \cup B(2; (\varepsilon, 2, \varepsilon); 3_a) \cup B(2; (\varepsilon, 2, \varepsilon); 3_b) \\ B(3_a) &= B(3_a; (12, 3_a, 12); 1) \\ B(3_b) &= B(3_b; (1, 3_b, 2); 2). \end{aligned}$$

The new system verifies conditions (13) and (14), then the Theorem 3.2 proves that \mathcal{F} , \mathcal{F}_1 and \mathcal{F}_2 are connected, but \mathcal{F}_3 is the union of two connected components, $\mathcal{F}_3 = \mathcal{F}_{3_a} \cup \mathcal{F}_{3_b}$. ■

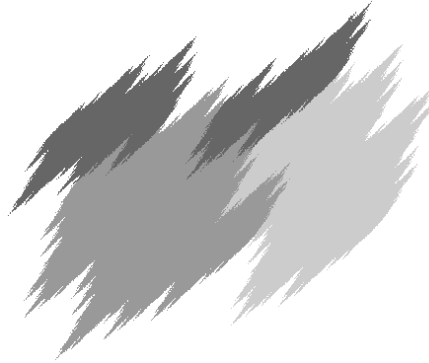


Figure 4: Representation of $\sigma : 1 \mapsto 12312, 2 \mapsto 132, 3 \mapsto 2$

Remark 3.1. We give in the proof of the preceding theorem a new alphabet $\overline{\mathcal{A}}$ and a new substitution $\overline{\sigma}$ verifying $\pi(\overline{\sigma}(i_k)) = \sigma(i)$. We naturally pose $\delta(i_k) = \delta(i)$. This gives a way to geometrically represent some non Pisot type substitutions, mainly non one-to-one substitutions, like $\overline{\sigma}$ given in the preceding example.

4 Conclusion

The Theorem 3.2 can be used to give a new proof the well-known result that the geometric representation of a primitive Sturmian substitution is a two intervals exchange, and the new result that the geometric representation of a primitive Arnoux-Rauzy substitution is connected (see [Can]), which are in some way generalized Sturmian substitutions.

Acknowledgements The author thanks Pascal Hubert for stimulating discussions on the subject and the referee for useful suggestions.

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