INDICATORS OF POINTED HOPF ALGEBRAS OF DIMENSION pq OVER CHARACTERISTIC p

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Abstract. Let p, q be two distinct primes. We consider pointed Hopf algebras of dimension pq over an algebraically closed field of characteristic p. We compute higher Frobenius-Schur indicators of these Hopf algebras through the associated graded Hopf algebras with respect to their coradical filtrations. The resulting indicators are gauge invariants for the monoidal representation categories of these algebras.

1. Introduction

Higher Frobenius-Schur (FS) indicators of Hopf algebras were introduced in [4] and their properties were further studied in [2, 3, 6]. These indicators provide gauge invariants of the monoidal representation categories of the underlying Hopf algebras, see [2, Theorem 2.2]. The sequences of higher FS-indicators of certain Hopf algebras have been computed in, e.g., [1, 2, 9]. In this article, we are concerned with the FS-indicators of pointed Hopf algebras of dimension pq over a field of characteristic p. The classification of such Hopf algebras was recently obtained in [8]. We define the arithmetic function χ_r, associated to a prime r, for positive integers n by

$$\chi_r(n) = \begin{cases} r, & \text{if} \quad r \mid n; \\ 1, & \text{if} \quad r \nmid n. \end{cases}$$

Then, our main result is the following theorem.

Theorem 1.1. *Let p, q be distinct primes, and H be a pq-dimensional pointed Hopf algebra over an algebraically closed field of characteristic p. If H is commutative and co-commutative, then the nth indicator*

$$\nu_n(H) \equiv \chi_p(n)\chi_q(n) \pmod{p}.$$

Otherwise,

$$\nu_n(H) \equiv \chi_p(n) \pmod{p}.$$

We first introduce some notation. The proof of the main result is in Section 2. Readers unfamiliar with Hopf algebras are referred to standard textbooks such as [5, 7]. Throughout, p, q are distinct primes, and H is a finite-dimensional Hopf algebra over an algebraically closed base field $\mathbf{k}$ of characteristic p. We use the standard notation $(H, m, u, \Delta, \varepsilon, S)$, where $m : H \otimes H \to H$ is the multiplication map, $u : \mathbf{k} \to H$ is the unit map, $\Delta : H \to H \otimes H$ is the comultiplication map, $\varepsilon : H \to \mathbf{k}$ is the counit map, and $S : H \to H$ is the antipode. The vector space dual of H is also a Hopf algebra and will be denoted by H^*. We use the Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. If $f, g \in H^*$, then $fg(h) = \sum f\left(h_{(1)}\right) g\left(h_{(2)}\right)$ for any $h \in H$ and $\varepsilon_{H^*}(f) = f(1)$.

Let n be a positive integer. Suppose $h_1, \dots, h_n \in H$. Then the nth *power of multiplication* is defined as $m^{[n]}(h_1 \otimes \cdots \otimes h_n) = h_1 \cdots h_n$. Let $h \in H$. The nth power of comultiplication is defined to be

$$\Delta^{[n]}(h) = \begin{cases} h, & n = 1; \\ (\Delta^{[n-1]} \otimes \mathrm{id})\,(\Delta(h))\,, & n \geq 2. \end{cases}$$

The nth *Sweedler power* of h is defined to be

$$P_n(h) = h^{[n]} = \begin{cases} \epsilon(h) 1_H, & n = 0; \\ m^{[n]} \circ \Delta^{[n]}(h), & n \geq 1. \end{cases}$$

Then the nth *indicator* [2, Definition 2.1] of H is given by

$$\nu_n(H) = \mathrm{Tr}\big(S \circ P_{n-1}\big).$$

In particular, $\nu_1(H) = 1$ and $\nu_2(H) = \mathrm{Tr}(S)$.

2. The Proof of the Main Result

The classification of pointed Hopf algebras of dimension pq was recently obtained by Xiong [8]. The following theorem is a restatement of Xiong's result, with slightly different notation. For one of the noncommutative cases, we use the relation $xg - \xi gx$ instead of $gx - \xi^{-1}xg$ that appeared in the original form of this result.

Theorem 2.1. [8, Theorem 2.19] *Let ξ be a qth primitive root of unity and $\delta = 0$ or 1. Then a pq-dimensional pointed Hopf algebra H is isomorphic to one of the following:*

1) $\mathbf{k}[g, x]/(g^q - 1,\ x^p - \delta x)$,
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + 1 \otimes x$;

2) $\begin{cases} \mathbf{k}\langle g, x\rangle/(g^q - 1,\ x^p,\ xg - \xi gx), & q \nmid p-1, \\ \mathbf{k}\langle g, x\rangle/(g^q - 1,\ x^p - \delta x,\ xg - \xi gx), & q \mid p-1, \end{cases}$
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + 1 \otimes x$;

3) $\mathbf{k}[g,\, x]/(g^q - 1,\ x^p - \delta(1 - g^p))$,
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x$;

4) $\begin{cases} \mathbf{k}\langle g,\, x\rangle/(g^q - 1,\ x^p - x,\ gx - xg - g + g^2), & q \nmid p - 1, \\ \mathbf{k}[g,\, x]/(g^q - 1,\ x^p - x), & q \mid p - 1, \end{cases}$
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x$.

5) *A group algebra kG with $|G| = pq$.*

The approach of our proof for Theorem 1.1 is based on the above classification result and the following theorem and proposition. We also provide a lemma prior to the proof of Theorem 1.1. A *left integral* in a finite-dimensional Hopf algebra H is an element $\Lambda \in H$ such that $h\Lambda = \varepsilon(h)\Lambda$, for all $h \in H$. The span of the left integrals in H is a one-dimensional space.

Theorem 2.2. [2, Corollary 2.6] *Suppose $\lambda \in H^*$ and $\Lambda \in H$ are both left integrals such that $\lambda(\Lambda) = 1$. Then $\nu_n(H) = \lambda\left(\Lambda^{[n]}\right)$ for all positive integers n.*

Proposition 2.3. [6, Corollary 3.17, Corollary 4.6] *Let H and H' be finite-dimensional Hopf algebras. Then, for all positive integers n, we have the following:*

1) *$\nu_n(H^*) = \nu_n(H)$ and $\nu_n(H \otimes H') = \nu_n(H) \cdot \nu_n(H')$.*
2) *If H is filtered, then $\nu_n(\mathrm{gr}H) = \nu_n(H)$.*
3) *$\{\nu_n(H)\}$ is periodic.*

Although Proposition 2.3, 2) is true for any Hopf filtration, in this paper we use the coradical filtration [5, Section 5.2].

In the next lemma, we use the multinomial coefficients

$$\binom{m}{k_1, \ldots, k_n} := \frac{(m!)}{(k_1!) \cdots (k_n!)},$$

where $k_1, \ldots, k_n$ are nonnegative integers such that $\sum_{i=1}^{n} k_i = m$.

Lemma 2.4. *Let i, n be integers for $0 \le i \le q - 1$ and $n \ge 1$. Suppose nonnegative integers $k_1, \ldots, k_{n+1}$ form a partition of $p - 1$. Regarding the Hopf algebras in Theorem 2.1, we have*

1) *If $xg = gx$, $\Delta(g) = g \otimes g$, and $\Delta(x) = x \otimes 1 + g \otimes x$, then*

$$(g^i x^{p-1})^{[n+1]} = \sum_{k_1 + \cdots + k_{n+1} = p-1} \binom{p-1}{k_1, \ldots, k_{n+1}} g^{(n+1)i + k_1 + 2k_2 + \cdots + nk_n} x^{p-1}.$$

2) *If* $xg = \xi gx$, $\Delta(g) = g \otimes g$, *and* $\Delta(x) = x \otimes 1 + 1 \otimes x$, *then*

$$(g^i x^{p-1})^{[n+1]} = \left((\xi^i)^n + \cdots + \xi^i + 1\right)^{p-1} g^{(n+1)i} x^{p-1}.$$

Proof. 1) If $n = 1$, then $(g^i x^{p-1})^{[n+1]} = (g^i x^{p-1})^{[2]} = m \circ \Delta(g^i x^{p-1})$. First,

$$\Delta(g^i x^{p-1}) = g^i \otimes g^i (x \otimes 1 + g \otimes x)^{p-1} = \sum_{k_1=0}^{p-1} \binom{p-1}{k_1} g^i x^{p-1-k_1} g^{k_1} \otimes g^i x^{k_1}.$$

Note that $xg = gx$. Then

$$(g^i x^{p-1})^{[2]} = \sum_{k_1=0}^{p-1} \binom{p-1}{k_1} g^{2i+k_1} x^{p-1}.$$

Now let $n = 2$, and k_1, k_2, k_3 form a partition of $p-1$. Then

$$\begin{aligned}\Delta^{[3]}(g^i x^{p-1}) &= \Delta \otimes \mathrm{id} \circ \Delta(g^i x^{p-1}) \\ &= \sum_{k_1=0}^{p-1} \binom{p-1}{k_1} \Delta(g^i x^{p-1-k_1}) \Delta(g)^{k_1} \otimes \mathrm{id}(g^i x^{k_1}).\end{aligned}$$

Following the case when $n = 1$, we have

$$\Delta(g^i x^{p-1-k_1}) = \sum_{k_2=0}^{p-1-k_1} \binom{p-1-k_1}{k_2} g^i x^{k_3} g^{k_2} \otimes g^i x^{k_1}.$$

Hence,

$$\begin{aligned}&\Delta^{[3]}(g^i x^{p-1}) \\ &= \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-1-k_1} \binom{p-1}{k_1} \binom{p-1-k_1}{k_2} g^i x^{k_3} g^{k_1+k_2} \otimes g^i x^{k_2} g^{k_1} \otimes g^i x^{k_1}.\end{aligned}$$

Thus, using multinomial coefficients, we have

$$(g^i x^{p-1})^{[3]} = \sum_{k_1+k_2+k_3=p-1} \binom{p-1}{k_1, k_2, k_3} g^{3i+2k_1+k_2} x^{p-1}.$$

Since the formula is symmetric on the k_i's, for the sake of the notation, we will write

$$(g^i x^{p-1})^{[3]} = \sum_{k_1+k_2+k_3=p-1} \binom{p-1}{k_1, k_2, k_3} g^{3i+k_1+2k_2} x^{p-1}.$$

Finally, one can show inductively that

$$(g^i x^{p-1})^{[n+1]} = \sum_{k_1+\cdots+k_{n+1}=p-1} \binom{p-1}{k_1, \ldots, k_{n+1}} g^{(n+1)i+k_1+2k_2+\cdots+nk_n} x^{p-1}.$$

2) It can be shown that

$$\begin{aligned}\Delta^{[n+1]}(x^{p-1}) &= \big(\Delta^{[n+1]}(x)\big)^{p-1}\\ &= \big(x\otimes 1\otimes\cdots\otimes 1+\cdots+1\otimes\cdots\otimes 1\otimes x\big)^{p-1}\\ &= \sum_{k_1+\cdots+k_{n+1}=p-1}\binom{p-1}{k_1,\ldots,k_{n+1}}x^{k_1}\otimes x^{k_2}\otimes\cdots\otimes x^{k_{n+1}}.\end{aligned}$$

Note that we have $xg=\xi gx$. Hence,

$$(g^ix^{p-1})^{[n+1]} = \sum_{k_1+\cdots+k_{n+1}=p-1}\binom{p-1}{k_1,\ldots,k_{n+1}}g^ix^{k_1}g^ix^{k_2}\ldots g^ix^{k_{n+1}}$$

$$= \sum_{k_1+\cdots+k_{n+1}=p-1}\binom{p-1}{k_1,\ldots,k_{n+1}}(\xi^i)^{nk_1}(\xi^i)^{(n-1)k_2}\cdot(\xi^i)^{k_n}\cdot g^{(n+1)i}x^{p-1}$$

$$=\Big((\xi^i)^n+\cdots+\xi^i+1\Big)^{p-1}g^{(n+1)i}x^{p-1}.$$

□

The proof of Theorem 1.1. Let H be a pq-dimensional pointed Hopf algebra over $\mathbf{k}$ given in Theorem 2.1. We first compute the indicators of H, when H is a group algebra $\mathbf{k}G$. It follows from Appendix A in [8] that G is isomorphic to one of the following.

- $C_{pq}\cong C_p\times C_q$.
- $C_p\rtimes C_q=\langle g,h\,|\,g^q=h^p=1,\,ghg^{-1}=h^s\rangle$, where $p\equiv 1 \pmod q$ and s is a nontrivial solution of $s^q\equiv 1 \pmod p$.
- $C_q\rtimes C_p=\langle g,h\,|\,g^p=h^q=1,\,ghg^{-1}=h^t\rangle$, where $q\equiv 1 \pmod p$ and t is a nontrivial solution of $s^p\equiv 1 \pmod q$.

By the definition of indicator, it is straightforward to see that

$$\nu_n(\mathbf{k}G)=\mathrm{Tr}\,(S\circ P_{n-1}:\ \mathbf{k}G\to\mathbf{k}G)=\#\{g\in G\mid g^n=1_G\}.$$

Then, one can apply counting arguments to the three cases, when H is a group algebra. For example, if $H\cong\mathbf{k}(C_p\rtimes C_q)$, then

$$\#\{g\in C_p\rtimes C_q\,|\,g^n=1\}=\begin{cases}pq-p+1\equiv 1 \pmod p, & \text{for } p\nmid n,\ q\mid n;\\ p\equiv 0 \pmod p, & \text{for } p\mid n,\ q\nmid n;\\ pq\equiv 0 \pmod p, & \text{for } p\mid n,\ q\mid n;\\ 1\equiv 1 \pmod p, & \text{for } p\nmid n,\ q\nmid n.\end{cases}$$

Hence,

$$\nu_n\left(\mathbf{k}(C_p\rtimes C_q)\right)\equiv\chi_p(n)\pmod p.$$

Similarly, one can show that

$$\nu_n\left(\mathbf{k}(C_p \times C_q)\right) \equiv \chi_p(n)\chi_q(n) \pmod{p},$$
$$\nu_n\left(\mathbf{k}(C_q \rtimes C_p)\right) \equiv \chi_p(n) \pmod{p}.$$

Now, assume H is one of the Hopf algebras in 1)–4) of Theorem 2.1. The linear basis of H can be chosen as $\{g^i x^j \mid 0 \le i \le q-1, 0 \le j \le p-1\}$. The associated graded Hopf algebra gr$H$ is one of the following, in the correspondence 1) $\rightsquigarrow A$, 3), 4) $\rightsquigarrow B$, and 2) $\rightsquigarrow C$.

- $A = \mathbf{k}[g, x]/(g^q - 1,\ x^p)$, with $\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + 1 \otimes x$.
- $B = \mathbf{k}[g, x]/(g^q - 1,\ x^p)$, with $\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x$
- $C = \mathbf{k}\langle g, x\rangle/(g^q - 1,\ x^p,\ xg - \xi gx)$, with $\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + 1 \otimes x$.

Next, we find case by case the indicators of these associated graded Hopf algebras.

Case *A*. Note that $A = \mathbf{k}[g]/(g^q - 1) \otimes \mathbf{k}[x]/(x^p)$, with $\Delta(g) = g \otimes g$, and $\Delta(x) = x \otimes 1 + 1 \otimes x$. Following from the counting argument,

$$\nu_n(\mathbf{k}[g]/(g^q - 1)) = \#\{h \in \langle g\rangle \mid h^n = 1_{\langle g\rangle}\},$$

we have

$$\nu_n\big(\mathbf{k}[g]/(g^q - 1)\big) \equiv \chi_q(n) \pmod{p}, \quad \text{for all} \quad n \ge 1.$$

By [9, Corollary 3.9], it follows that

$$\nu_n\big(\mathbf{k}[x]/(x^p)\big) \equiv \chi_p(n) \pmod{p}, \quad \text{for all} \quad n \ge 1.$$

Therefore,

$$\nu_n(A) \equiv \chi_p(n)\chi_q(n) \pmod{p}, \quad \text{for all} \quad n \ge 1.$$

Case *B*. The comultiplication on the basis elements is given by

$$\Delta(g^i x^j) = \sum_{k=0}^{j} \binom{j}{k} g^{k+i} x^{j-k} \otimes g^i x^k.$$

Let $\lambda_{B^*} = \delta_{g^{1-p}x^{p-1}} \in B^*$. Then

$$\delta_{g^{1-p}x^{p-1}}(g^i x^j) = \begin{cases} 1, & \text{if } j = p-1 \text{ and } i \equiv 1-p \pmod{q}; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand,

$$\sum_{k=0}^{j} \binom{j}{k} g^{k+i} x^{j-k} \cdot \delta_{g^{1-p}x^{p-1}}(g^i x^k)$$
$$= \begin{cases} g^{p-1+1-p}x^0 = 1, & \text{if } j = k = p-1 \text{ and } i \equiv 1-p \pmod{q}; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, by [1, Lemma 2.3], $\lambda_{B^*} = \delta_{g^{1-p}x^{p-1}}$ is a left integral of B^*. Next, consider the element $\Lambda_B = (1 + g + \cdots + g^{q-1})x^{p-1} \in B$. It is clear that $x\Lambda_B = 0 = \varepsilon(x)\Lambda_B$ and $g\Lambda_B = \Lambda_B = \varepsilon(g)\Lambda_B$. Hence, $\Lambda_B = (1+g+\cdots+g^{q-1})x^{p-1}$ is a left integral of B. It is clear that $\lambda_{B^*}(\Lambda_B) = 1$. Following Theorem 2.2, for $n \geq 1$ we compute

$$\nu_{n+1}(B) = \lambda_{B^*}\left(\Lambda_B^{[n+1]}\right) = \delta_{g^{1-p}x^{p-1}}\Big(\sum_{i=0}^{q-1}(g^i x^{p-1})^{[n+1]}\Big).$$

Note that $gx = xg$ and $\delta(x) = x \otimes 1 + g \otimes x$. By Lemma 2.4,

$$\nu_{n+1}(B) = \sum_{\substack{0\leq i\leq q-1\\ k_1+\cdots+k_{n+1}=p-1}} \binom{p-1}{k_1, \ldots, k_{n+1}} \delta_{g^{1-p}}\left(g^{(n+1)i+k_1+2k_2+\cdots+nk_n}\right).$$

For any fixed nonnegative partition $k_1, \ldots, k_{n+1}$ of $p-1$ we have

$$\sum_{i=0}^{q-1} \delta_{g^{1-p}}\left(g^{(n+1)i+k_1+2k_2+\cdots+nk_n}\right) = \begin{cases} 1, & \text{if } q \nmid n+1; \\ q\delta_{g^{1-p}}\left(g^{k_1+2k_2+\cdots+nk_n}\right), & \text{if } q \mid n+1. \end{cases}$$

Hence, for $q \nmid n+1$, we have

$$\nu_{n+1}(B) = \sum_{k_1+\cdots+k_{n+1}=p-1} \binom{p-1}{k_1, \ldots, k_{n+1}} = (n+1)^{p-1}.$$

For $q \mid n+1$, we have

$$\nu_{n+1}(B) = \sum_{\substack{k_1+\cdots+k_{n+1}=p-1\\ k_1+2k_2+\cdots+nk_n\equiv 1-p \pmod q}} q\binom{p-1}{k_1, \ldots, k_{n+1}}.$$

Case *C*. Since $\Delta(x) = x \otimes 1 + 1 \otimes x$, we have

$$\Delta(g^i x^j) = \sum_{k=0}^{j} \binom{j}{k} g^i x^{j-k} \otimes g^i x^k,$$

for the basis elements $g^i x^j$ for $0 \leq i \leq q-1$ and $0 \leq j \leq p-1$. By a similar argument as that for B, the element $\lambda_{C^*} = \delta_{x^{p-1}} \in C^*$ is a left integral of C^*. Next, consider the element $\Lambda_C = (1 + g + \cdots + g^{q-1})x^{p-1} \in C$. It is clear that $g\Lambda_C = \varepsilon(g)\Lambda_C$. Note that $xg = \xi gx$. Then

$$x\Lambda_C = x(1+g+\cdots+g^{q-1})x^{p-1} = \left(1+\xi g+\cdots+(\xi g)^{q-1}\right)x^p = 0 = \varepsilon(x)\Lambda_C.$$

Hence, $\Lambda_C = (1 + g + \cdots + g^{q-1})x^{p-1}$ is a left integral of C. Again, by Theorem 2.2 and Lemma 2.4,

$$\begin{aligned}\nu_{n+1}(C) &= \delta_{x^{p-1}}\left(\sum_{i=0}^{q-1}(g^i x^{p-1})^{[n+1]}\right)\\ &= \delta_{x^{p-1}}\left(\sum_{i=0}^{q-1}\left((\xi^i)^n + \cdots + \xi^i + 1\right)^{p-1} g^{(n+1)i}x^{p-1}\right)\\ &= \delta_{x^{p-1}}\left((n+1)^{p-1}x^{p-1} + \sum_{i=1}^{q-1}\left(\frac{(\xi^i)^{n+1}-1}{\xi^i - 1}\right)^{p-1} g^{(n+1)i}x^{p-1}\right)\\ &\equiv (n+1)^{p-1} \pmod{p}.\end{aligned}$$

It follows from Fermat's Little Theorem that $(n)^{p-1} \equiv \chi_p(n) \pmod{p}$. Hence, we have $\nu_n(C) \equiv \chi_p(n) \pmod{p}$ for any $n \geq 1$.

Note that A is self-dual, while B and C are dual Hopf algebras. Hence, we have $\nu_n(B) = \nu_n(C)$. To see $B^* \cong C$ as Hopf algebras, we consider the linear functionals X and G on B such that $X(g^i x^j) = \delta_{1,j}$ and $G(g^i x^j) = \delta_{0,j}(1/\xi)^i$. One can check that X is an $(\varepsilon, \varepsilon)$-derivation of B (meaning $X(ab) = \varepsilon(a)X(b) + \varepsilon(b)X(a)$ for any $a, b \in B$) and that G is an algebra map from B to $\mathbf{k}$. By duality, we know X is a primitive element and G is a group-like element in B^*. Then, it is straightforward to check that the map $B^* \to C$ given by $X \mapsto x$ and $G \mapsto g$ is an isomorphism of Hopf algebras.

In summary, the indicators for the non-group algebra cases 1)–4) can be obtained by combining Proposition 2.3 and the three cases of graded Hopf algebras. By adding the group cases shown in the beginning, we prove Theorem 1.1. Moreover, the fact that $\nu_n(B) = \nu_n(C)$ gives the following corollary.

Corollary 2.5. *Let p, q be two distinct primes, $n > 1$ be an integer, and $q \mid n$. The following combinatorics identity holds*

$$\sum_{\substack{k_1+\cdots+k_n=p-1\\ k_1+2k_2+\cdots+(n-1)k_{n-1}\equiv 1-p \pmod{q}}} q\binom{p-1}{k_1,\ldots,k_n} \equiv n^{p-1} \pmod{p}.$$

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