ON CONSTRUCTING CHAOTIC MAPS WITH A PRESCRIBED PROBABILITY DISTRIBUTION

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Abstract. In this paper we discuss how to construct piecewise linear chaotic maps with a prescribed probability distribution on a finite number of open intervals of equal length that form a partition of the unit interval. The idea and method of how to find such a map are given in [3]. But a formal proof is not given. In this paper we provide a formal proof.

1. Problem

In this paper we consider the following problem. We divide the unit interval $[0,1]$ into n open subintervals $\{I_j\}_{j=1}^n = \{(\frac{j-1}{n}, \frac{j}{n})\}_{j=1}^n$ of equal length. Suppose that $p = [p_1 \dots p_n]^T$ is a given positive probability vector, that is, $\sum_{j=1}^n p_j = 1$ and each $p_j > 0$. Now, the problem is to find a piecewise linear map $\tau : [0,1] \to [0,1]$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{I_j}(\tau^i(x)) = p_j \quad m - \text{a.e} \quad \text{for} \quad 1 \le j \le n, \tag{1.1}$$

where χ_{I_j} is the characteristic map over I_j for each j, τ^i denotes the composition of τ with itself i times, and m is the Lebesgue measure. The idea and method of how to find such a map τ are given in [3] (see also [4] and [5]). But a formal proof has not been given. In this paper we provide a formal proof.

2. Preliminaries

Throughout the paper we assume that $\mathcal{B}$ is the Borel σ-algebra on the closed unit interval $[0,1]$. Every measure mentioned in this paper is defined on $\mathcal{B}$. We need three definitions before we state the celebrated Birkhoff Ergodic Theorem.

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Definition 1. *The map* $\tau : [0,1] \to [0,1]$ *is called measurable if* $\tau^{-1}(\mathcal{B}) \subseteq \mathcal{B}$, *that is,* $B \in \mathcal{B}$ *implies* $\tau^{-1}(B) \in \mathcal{B}$, *where* $\tau^{-1}(B) = \{x \in [0,1] : \tau(x) \in B\}$.

Definition 2. *We say that the measurable map* $\tau : [0,1] \to [0,1]$ *preserves measure* μ *or that measure* μ *is* τ*-invariant if* $\mu(\tau^{-1}(B)) = \mu(B)$ *for all* $B \in \mathcal{B}$.

Definition 3. *Let* $\tau : [0,1] \to [0,1]$ *be a measurable map. A set* $B \in \mathcal{B}$ *is said to be an invariant set of* τ *if* $\tau^{-1}(B) = B$. *The map* τ *is said to be ergodic w.r.t. the measure* μ *if whenever* $B \in \mathcal{B}$ *is an invariant set of* τ, *then* $\mu(B) = 0$ *or* $\mu(B^c) = 0$.

Now we are ready to recall the Birkhoff Ergodic Theorem.

The Birkhoff Ergodic Theorem. *Suppose* $\tau : [0,1] \to [0,1]$ *is measurable and the probability measure* μ *is* τ*-invariant. Then for any* $f \in L^1([0,1], \mathcal{B}, \mu)$, *there exists a function* $\hat{f} \in L^1([0,1], \mathcal{B}, \mu)$ *such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = \hat{f}(x), \quad \mu - \text{a.e.} \tag{2.1}$$

Furthermore,

$$\hat{f} \circ \tau = \hat{f}, \quad \mu - \text{a.e.} \tag{2.2}$$

and

$$\int_0^1 \hat{f}\ d\mu = \int_0^1 f\ d\mu. \tag{2.3}$$

In addition, if τ *is ergodic w.r.t. the probability measure* μ, *then (2.2) implies that* $\hat{f}$ *is constant* μ*-a.e., so using (2.3) and the fact that* μ *is a probability measure we have*

$$\hat{f} = \int_0^1 f\ d\mu, \quad \mu - \text{a.e.}$$

Thus, (2.1) becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = \int_0^1 f\ d\mu, \quad \mu - \text{a.e.} \tag{2.4}$$

Proof. See, for example, [2, Theorems 4.2.3 and 4.2.4]. □

Now we need two more definitions before we state Theorem 1.

Definition 4. *A measurable map* $\tau : [0,1] \to [0,1]$ *is nonsingular w.r.t. the Lebesgue measure* m *if* $m(B) = 0$ *implies* $m(\tau^{-1}(B)) = 0$, *where* $B \in \mathcal{B}$.

Definition 5. *If $\tau : [0,1] \to [0,1]$ is a nonsingular map w.r.t. the Lebesgue measure m, the linear operator $P_\tau : L^1([0,1],\mathcal{B},m) \to L^1([0,1],\mathcal{B},m)$ defined (implicitly) for any $g \in L^1([0,1],\mathcal{B},m)$ by*

$$\int_B P_\tau g \; dm = \int_{\tau^{-1}(B)} g \; dm \quad \text{for every} \quad B \in \mathcal{B}$$

is called the Frobenius-Perron operator associated with τ.

Remark 1. *For a given $g \in L^1([0,1],\mathcal{B},m)$, by choosing $B = [0,x]$, we see that*

$$\int_0^x P_\tau g \; dm = \int_{\tau^{-1}([0,x])} g \; dm \quad \text{for every} \quad x \in [0,1].$$

By differentiating both sides w.r.t. x, we have

$$\left(P_\tau g\right)(x) = \frac{d}{dx}\int_{\tau^{-1}([0,x])} g \; dm \quad \text{for } m - \text{a.e.} \quad x \in [0,1], \tag{2.5}$$

which is the explicit formula of the Frobenius-Perron operator associated with τ.

We have the following theorem that tells us the usefulness of the Frobenius-Perron operator associated with a nonsingular map $\tau : [0,1] \to [0,1]$ w.r.t. the Lebesgue measure m.

Theorem 1. *Let $\tau : [0,1] \to [0,1]$ be a nonsingular map w.r.t. the Lebesgue measure m, and let P_τ be the Frobenius-Perron operator associated with τ. Let $g \in L^1[(0,1],\mathcal{B},m)$ be a density function. Then the probability measure μ defined by*

$$\mu(B) = \int_B g \; dm \quad \text{for every} \quad B \in \mathcal{B} \tag{2.6}$$

is τ-invariant if and only if g is an invariant density under P_τ, that is, $P_\tau(g) = g$.

Proof. See, for example, [1, Proposition 4.2.7]. □

Now we need another basic definition before we state Theorem 2.

Definition 6. *The measure μ_1 is said to be an absolutely continuous measure w.r.t. the measure μ_2, denoted by $\mu_1 << \mu_2$, if $B \in \mathcal{B}$ and $\mu_2(B) = 0$, then $\mu_1(B) = 0$.*

Now we are ready to state Theorem 2.

Theorem 2. *Let $\tau : [0,1] \to [0,1]$ be nonsingular and ergodic w.r.t. the Lebesgue measure m. Suppose that $g \in L^1([0,1],\mathcal{B},m)$ is a density such that $P_\tau(g) = g$ and $g > 0$ on $[0,1]$. If we define the probability measure μ as in (2.6), then μ is τ-invariant, $\mu << m$, and τ is ergodic w.r.t. the probability measure μ. Furthermore, we also have $m << \mu$.*

Proof. Since $P_\tau(g) = g$ and the probability measure μ is defined as in (2.6), by Theorem 1 the probability measure μ is τ-invariant. By the definition of μ, we have $\mu << m$. So τ, which is ergodic w.r.t. the Lebesgue measure m, is also ergodic w.r.t. the probability measure μ. Furthermore, since $g > 0$ on $[0, 1]$, it is easy to see that we also have $m << \mu$. □

3. Construction of the Map τ

We define $g^* \in L^1([0, 1], \mathcal{B}, m)$ by

$$g^* = \sum_{j=1}^{n} (np_j)\chi_{I_j}, \tag{3.1}$$

where p_j is the jth component of the positive probability vector p given in Section 1. Note that g^* is not defined at $\{\frac{j}{n}\}_{j=0}^{n}$. But since the Lebesgue measure of this set is zero, it does not matter. Since p is a positive probability vector, g^* is a positive density function.

Suppose that we can find $\tau : [0, 1] \to [0, 1]$ such that

(i) τ is nonsingular and ergodic w.r.t. the Lebesgue measure m;
(ii) g^* is an invariant density of P_τ, that is, $P_\tau(g^*) = g^*$.

Let the probability measure μ be defined by (2.6) using g^* in place of g. Then by Theorem 2, the equation (2.4) in the Birkhoff Ergodic Theorem is valid for τ with any $f \in L^1([0, 1], \mathcal{B}, \mu)$. So using (2.4) with $f = \chi_{I_j}$, we obtain

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{I_j}(\tau^i(x)) &= \int_0^1 \chi_{I_j}\ d\mu = \int_{I_j} d\mu \\ &= \int_{I_j} g^*\ dm = p_j \quad \mu - \text{a.e.} \quad \text{for} \quad 1 \le j \le n.\end{aligned}$$

But since $m << \mu$ again by Theorem 2, the above equation can be written

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{I_j}(\tau^i(x)) = p_j \quad m - \text{a.e} \quad \text{for} \quad 1 \le j \le n,$$

which is (1.1). So all we have to do is to find τ fulfilling the conditions (i) and (ii).

The key of constructing such a τ is to construct a special column stochastic matrix. In [3] (see also [4] and [5]) the authors considered the following matrix

$$A_\beta = \beta I + (1 - \beta)pe^T \quad \text{with} \quad e^T = [1\ 1 \cdots 1], \tag{3.2}$$

where the vector p is the positive probability vector given in Section 1 and β is any real number in the interval $[0,1)$. We let $a_{ij}^{(\beta)}$ denote the (i,j)-entry of the matrix A_β for each $\beta \in [0,1)$. Now we collect two important properties of the matrix A_β for each $\beta \in [0,1)$.

Lemma 1. *The matrix A_β is a positive column stochastic matrix for every $\beta \in [0,1)$.*

Proof. Note that from (3.2), $a_{ii}^{(\beta)} = \beta + (1-\beta)p_i$ for $1 \le i \le n$ and $a_{ij} = (1-\beta)p_i$ for $1 \le i \ne j \le n$. Since $\beta \in [0,1)$ and p is a positive probability vector, we see that $a_{ij}^{(\beta)} > 0$ for $1 \le i,j \le n$. So A_β is a positive matrix for every $\beta \in [0,1)$.

To show that A_β is column stochastic, it is enough to show $e^T A_\beta = e^T$. Indeed, for every $\beta \in [0,1)$, using the fact that $e^T p = 1$, we have

$$\begin{aligned} e^T A_\beta &= e^T[\beta I + (1-\beta)p\, e^T] = \beta e^T + (1-\beta)(e^T p)e^T \\ &= \beta e^T + (1-\beta)e^T = e^T, \end{aligned}$$

as desired. □

Lemma 2. *The positive probability vector p is an eigenvector of A_β corresponding to eigenvalue 1 for every $\beta \in [0,1)$, that is $A_\beta p = p$ for every $\beta \in [0,1)$.*

Proof. We simply note that, for any $\beta \in [0,1)$, we have

$$\begin{aligned} A_\beta p &= [\beta I + (1-\beta)p\, e^T]p = \beta p + (1-\beta)p\,(e^T p) \\ &= \beta p + (1-\beta)p = p. \end{aligned}$$

□

Now we are ready to construct $\tau : [0,1] \to [0,1]$ which fulfills conditions (i) and (ii). In fact, we will construct a one parameter family $\{\tau_\beta\}_{\beta\in[0,1)}$ which fulfills conditions (i) and (ii) for every $\beta \in [0,1)$.

Now we construct such a family. Let β be an arbitrary number in the interval $[0,1)$ but fixed. First of all, we construct a partition $\{x_s^{(\beta)}\}_{s=0}^{n^2}$ of $[0,1]$ such that $x_0^{(\beta)} = 0$ and

$$x_{n(j-1)+k}^{(\beta)} = \frac{j-1}{n} + \frac{1}{n}\sum_{i=1}^{k} a_{ij}^{(\beta)}, \quad 1 \le j,k \le n.$$

Since by Lemma 1, $a_{ij}^{(\beta)} > 0$, $\{x_s^{(\beta)}\}_{s=0}^{n^2}$ is made of distinct points. Since again by Lemma 1, A_β is a column stochastic matrix, we have $x_{n(j-1)+n}^{(\beta)} = x_{nj}^{(\beta)} = \frac{j}{n}$ for $1 \le j \le n$. In particular, $x_{n^2}^{(\beta)} = 1$.

Now we define τ_β by defining $\tau_\beta|_{(x_{i-1}^{(\beta)},x_i^{(\beta)})}$ for $1 \le i \le n^2$ as follows: for $1 \le i \le n^2$ we let

$$\tau_\beta|_{(x_{i-1}^{(\beta)},x_i^{(\beta)})}(x) = \frac{1}{n(x_i^{(\beta)} - x_{i-1}^{(\beta)})}(x - x_{i-1}^{(\beta)}) + \frac{\mathrm{rem}(i-1,n)}{n}, \;\; x \in (x_{i-1}^{(\beta)}, x_i^{(\beta)}),$$

where $\mathrm{rem}(i-1,n)$ denotes the remainder when $i-1$ is divided by n. Note that τ_β is not defined at $\{x_s^{(\beta)}\}_{s=0}^{n^2}$. Since the Lebesgue measure of this set is zero, it does not matter. However, it is convenient to define $\tau_\beta(x_0^{(\beta)}) = 0$ and $\tau_\beta(x_{n(j-1)+k}^{(\beta)}) = \frac{k}{n}$ for $1 \le j,k \le n$ so that $\tau_\beta|_{I_j}$ is a piecewise linear strictly increasing continuous function onto the open interval $(0,1)$ for $1 \le j \le n$.

So if we let $\tau_\beta^{(j)} : I_j \to (0,1)$ be the restriction of τ_β to I_j for $1 \le j \le n$, then each $\tau_\beta^{(j)}$ is a one-to-one and onto function from I_j to $(0,1)$. Thus, if we let $g_\beta^{(j)} = \left(\tau_\beta^{(j)}\right)^{-1} : (0,1) \to I_j$ for $1 \le j \le n$, we have

$$\tau_\beta'\left(g_\beta^{(j)}(x)\right)\left(g_\beta^{(j)}\right)'(x) = 1,$$

because $\tau_\beta\left(g_\beta^{(j)}(x)\right) = x$, and hence,

$$\left(g_\beta^{(j)}\right)'(x) = \frac{1}{\tau_\beta'\left(g_\beta^{(j)}(x)\right)}. \tag{3.3}$$

One can check that

$$\tau_\beta'\left(g_\beta^{(j)}(x)\right) = \frac{1}{a_{ij}^{(\beta)}} \quad \text{when} \quad x \in I_i \quad \text{for} \quad 1 \le i,j \le n. \tag{3.4}$$

Now we are going to show that $P_{\tau_\beta}(g^*) = g^*$. Since $g^* = \sum_{j=1}^n (np_j)\chi_{I_j}$ by (3.1), we first find the expression of $P_{\tau_\beta}(\chi_{I_j})$. Note that by using (2.5), (3.3), and (3.4), we have

$$\begin{aligned} P_{\tau_\beta}(\chi_{I_j})(x) &= \frac{d}{dx}\int_{\tau_\beta^{-1}([0,x])} \chi_{I_j}(y)\, dm(y) = \frac{d}{dx}\int_{\tau_\beta^{-1}([0,x])\cap I_j} 1\, dm(y) \\ &= \frac{d}{dx}\int_{\frac{j-1}{n}}^{g_\beta^{(j)}(x)} 1\, dm(y) = \left(g_\beta^{(j)}\right)'(x) \\ &= \frac{1}{\tau_\beta'\left(g_\beta^{(j)}(x)\right)} = \sum_{i=1}^n a_{ij}^{(\beta)}\chi_{I_i}(x). \end{aligned}$$

In short, we have

$$P_{\tau_\beta}(\chi_{I_j}) = \sum_{i=1}^n a_{ij}^{(\beta)}\chi_{I_i}. \tag{3.5}$$

Now using (3.1), (3.5), and Lemma 2, we have

$$\begin{aligned} P_{\tau_\beta}(g^*) &= P_{\tau_\beta}\left(\sum_{j=1}^{n}(np_j)\chi_{I_j}\right) = \sum_{j=1}^{n}(np_j)P_{\tau_\beta}(\chi_{I_j}) \\ &= \sum_{j=1}^{n}(np_j)\sum_{i=1}^{n}a_{ij}^{(\beta)}\chi_{I_i} = n\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}^{(\beta)}p_j\right)\chi_{I_i} \\ &= n\sum_{i=1}^{n}[A_\beta p]_i\chi_{I_i} = n\sum_{i=1}^{n}p_i\chi_{I_i} \\ &= \sum_{i=1}^{n}(np_i)\chi_{I_i} = g^*, \end{aligned}$$

where $[A_\beta p]_i$ is the ith component of the vector $A_\beta p$, and hence, $P_{\tau_\beta}(g^*) = g^*$.

But this is true for every $\beta \in [0,1)$. So τ_β, for any $\beta \in [0,1)$, has g^* as the common invariant density of its corresponding Frobenius-Perron operator P_{τ_β}. From the construction of τ_β, it is clear that τ_β is nonsingular and ergodic w.r.t. the Lebesgue measure m for every $\beta \in [0,1)$. So $\{\tau_\beta\}_{\beta\in[0,1)}$ fulfills conditions (i) and (ii) for every $\beta \in [0,1)$. Thus, $\{\tau_\beta\}_{\beta\in[0,1)}$ is a one parameter family of solutions to the problem given in Section 1.

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