SOME OPERATORS IN IDEAL TOPOLOGICAL SPACES

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Abstract. In this paper, we give an extensive study of ideal topological spaces and introduce some new types of sets with the help of a local function. Several characterizations of these sets will also be discussed through this paper. Moreover, we obtain characterizations of Ψ_ω-operator and ω-codense.

1. Introduction and Preliminaries

A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [5] if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well-known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X, τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, that can be defined in the same way as the closure $Cl(A)$ and the interior $Int(A)$ of A in (X, τ), respectively, will be denoted by $Cl_\omega(A)$ and $Int_\omega(A)$, respectively. Several characterizations of ω-closed sets were provided in [1, 3].

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ is called the local function of A with respect to $\mathcal{I}$ and τ (see [4, 7]). We simply write A^* instead of $A^*(\mathcal{I}, \tau)$, in case there is no chance for confusion. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology $\tau^*(\mathcal{I})$, finer than τ, generated by the base the base $\beta(\mathcal{I},\tau)=\{U-J : U\in\tau \text{ and } J\in\mathcal{I}\}$. It is known in [4] that $\beta(\mathcal{I},\tau)$ is not always a topology. When there is no ambiguity, $\tau^*(\mathcal{I})$ is denoted by τ^*. Recall that A is said to be $*$-dense in itself (resp. τ^*-closed, $*$-perfect) if $A\subseteq A^*$ (resp. $A^*\subseteq A$, $A=A^*$). For a subset $A\subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of A in (X,τ^*), respectively. Let $(X,\tau,\mathcal{I})$ be an ideal topological space. We say the topology τ is compatible with the ideal $\mathcal{I}$, denoted $\tau\sim\mathcal{I}$, if the following holds: for every $A\subseteq X$, if for every $x\in A$ there exists a $U\in\tau(x)$ such that $U\cap A\in\mathcal{I}$, then $A\in\mathcal{I}$ [4].

Definition 1.1. [2] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. For a subset A of X, we define the following set: $A_\omega(\mathcal{I},\tau)=\{x\in X : A\cap U\notin\mathcal{I}$ for every $U\in\tau_\omega(x)\}$, where $\tau_\omega(x)=\{U\in\tau_w : x\in U\}$. In case there is no confusion, $A_\omega(\mathcal{I},\tau)$ is briefly denoted by A_ω and is called the ω-local function of A with respect to $\mathcal{I}$ and τ.*

Lemma 1.2. [2] *Let (X,τ) be an ideal topological space, $\mathcal{I}$ and $\mathcal{J}$ be ideals on X, and let A and B be subsets of X. Then the following properties hold:*

1. *If $A\subseteq B$, then $A_\omega\subseteq B_\omega$.*
2. *If $\mathcal{I}\subseteq\mathcal{J}$, then $A_\omega(\mathcal{I})\supseteq A_\omega(\mathcal{J})$.*
3. *$A_\omega=Cl_\omega(A_\omega)\subseteq Cl_\omega(A)$ and A_ω is ω-closed in (X,τ).*
4. *If $A\subseteq A_\omega$, then $A_\omega=Cl_\omega(A_\omega)=Cl_\omega(A)$.*
5. *If $A\in\mathcal{I}$, then $A_\omega=\phi$.*

Theorem 1.3. [2] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A and B any subsets of X. Then the following properties hold:*

1. *$(\phi)_\omega=\phi$.*
2. *$(A_\omega)_\omega\subseteq A_\omega$.*
3. *$A_\omega\cup B_\omega=(A\cup B)_\omega$.*

Corollary 1.4. [2] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A and B be subsets of X with $B\in\mathcal{I}$. Then $(A\cup B)_\omega=A_\omega=(A-B)_\omega$.*

Remark 1.5. *In* [2]*, Al-Omari and Al-Saadi obtained that $Cl^*_\omega(A)=A\cup A_\omega$ is a Kuratowski closure operator. We will denote by τ^*_ω the topology generated by Cl^*_ω, that is, $\tau^*_\omega=\{U\subseteq X : Cl^*_\omega(X-U)=X-U\}$.*

Definition 1.6. [2] *Let (X,τ) be a topological space and $\mathcal{I}$ an ideal on X. A subset A of X is said to be τ^*_ω-closed if and only if $A_\omega\subseteq A$.*

Theorem 1.7. [2] *Let (X,τ) be a topological space and $\mathcal{I}$ an ideal on X. Then β is a basis for τ^*_ω, where $\beta(\mathcal{I},\tau)=\{V-I_0 : V\in\tau_\omega, I_0\in\mathcal{I}\}$.*

Definition 1.8. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then an ideal $\mathcal{I}$ is said to be ω-codense if $\tau_\omega\cap\mathcal{I}=\phi$.*

Theorem 1.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:*

(1) *$\mathcal{I}$ is ω-codense;*
(2) *If $I \in \mathcal{I}$, then $Int_\omega(I) = \phi$;*
(3) *For every $G \in \tau_\omega$, $G \subseteq G_\omega$;*
(4) *$X = X_\omega$.*

Proof.
(1) $\Rightarrow$ (2): Let $\mathcal{I}$ be ω-codense and $I \in \mathcal{I}$. Suppose that $x \in Int_\omega(I)$. Then there exists $U \in \tau_\omega$ such that $x \in U \subseteq I$. Since $I \in \mathcal{I}$, it follows that $\phi \neq \{x\} \subseteq U \in \tau_\omega \cap \mathcal{I}$. This contradicts $\tau_\omega \cap \mathcal{I} = \phi$. Therefore, $Int_\omega(I) = \phi$.
(2) $\Rightarrow$ (3): Let $x \in G$. Suppose that $x \notin G_\omega$. Then there exists $U_x \in \tau_\omega(x)$ such that $G \cap U_x \in \mathcal{I}$. By (2), $x \in G \cap U_x = Int_\omega(G \cap U_x) = \phi$. Hence, $x \in G_\omega$ and $G \subseteq G_\omega$
(3) $\Rightarrow$ (4): This follows since X is ω-open so $X = X_\omega$.
(4) $\Rightarrow$ (1):
$X = X_\omega = \{x \in X : U \cap X = U \notin \mathcal{I}$ for each ω-open set U containing $x\}$. Hence, $\tau_\omega \cap \mathcal{I} = \phi$. □

Definition 1.10. [2] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say the τ is ω-compatible with the ideal $\mathcal{I}$, denoted $\tau \sim_\omega \mathcal{I}$, if the following holds for every $A \subseteq X$; if for every $x \in A$ there exists $U \in \tau_\omega(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.*

Lemma 1.11. [2] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, then $\tau \sim_\omega \mathcal{I}$ if and only if $A - A_\omega \in \mathcal{I}$ for every $A \subseteq X$.*

Lemma 1.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is ω-open set, then it is ω-codense if and only if $A_\omega = Cl_\omega(A)$.*

Proof. Let A be ω-codense and A be a nonempty ω-open set. Then by Lemma 1.2 (3), we have $A_\omega \subseteq Cl_\omega(A)$. Let $x \in Cl_\omega(A)$. Then for all ω-open set U_x containing x, we have $U_x \cap A \neq \phi$. Again $U_x \cap A$ is a nonempty ω-open set, so $U_x \cap A \notin \mathcal{I}$, since $\mathcal{I}$ is ω-codense. Hence, $x \in A_\omega$. Therefore, $A_\omega = Cl_\omega(A)$. Conversely, for an ω-open set A, we have $A_\omega = Cl_\omega(A)$. Then $X = X_\omega$ and this implies that A is ω-codense by Theorem 1.9. □

Theorem 1.13. [2] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, τ be ω-compatible with $\mathcal{I}$, and $\tau_\omega \cap \mathcal{I} = \phi$. Let G be a τ^*_ω-open set such that $G = U - A$, where $U \in \tau_\omega$ and $A \in \mathcal{I}$. Then $Cl_\omega(G_\omega) = Cl_\omega(G) = G_\omega = U_\omega = Cl_\omega(U) = Cl_\omega(U_\omega)$.*

2. Ψ_ω-operator in Ideal Topological Space

Definition 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. An operator $\Psi_\omega : \mathcal{P}(X) \to \tau$ is defined as follows. For every $A \in X$, $\Psi_\omega(A) = \{x \in X :$ there exists $U \in \tau_\omega(x)$ such that $U - A \in \mathcal{I}\}$. Observe that $\Psi_\omega(A) = X - (X - A)_\omega$.*

Several basic facts concerning the behavior of the operator Ψ_ω are included in the following theorem.

Theorem 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties hold:*

(1) *If $A \subseteq X$, then $\Psi_\omega(A)$ is ω-open.*
(2) *If $A \subseteq B$, then $\Psi_\omega(A) \subseteq \Psi_\omega(B)$.*
(3) *If $A, B \in \mathcal{P}(X)$, then $\Psi_\omega(A \cap B) = \Psi_\omega(A) \cap \Psi_\omega(B)$.*
(4) *If $U \in \tau^*_\omega$, then $U \subseteq \Psi_\omega(U)$.*
(5) *If $A \subseteq X$, then $\Psi_\omega(A) \subseteq \Psi_\omega(\Psi_\omega(A))$.*
(6) *If $A \subseteq X$, then $\Psi_\omega(A) = \Psi_\omega(\Psi_\omega(A))$ if and only if $(X - A)_\omega = ((X - A)_\omega)_\omega$.*
(7) *If $A \in \mathcal{I}$, then $\Psi_\omega(A) = X - X_\omega$.*
(8) *If $A \subseteq X$, then $A \cap \Psi_\omega(A) = Int^*_\omega(A)$.*
(9) *If $A \subseteq X$, $I \in \mathcal{I}$, then $\Psi_\omega(A - I) = \Psi_\omega(A)$.*
(10) *If $A \subseteq X$, $I \in \mathcal{I}$, then $\Psi_\omega(A \cup I) = \Psi_\omega(A)$.*
(11) *If $(A - B) \cup (B - A) \in \mathcal{I}$, then $\Psi_\omega(A) = \Psi_\omega(B)$.*

Proof.

(1) This follows from Lemma 1.2 (3).

(2) This follows from Lemma 1.2 (1).

(3) It follows from (2) that $\Psi_\omega(A \cap B) \subseteq \Psi_\omega(A)$ and $\Psi_\omega(A \cap B) \subseteq \Psi_\omega(B)$. Hence, $\Psi_\omega(A \cap B) \subseteq \Psi_\omega(A) \cap \Psi_\omega(B)$. Now let $x \in \Psi_\omega(A) \cap \Psi_\omega(B)$. There exist $U, V \in \tau_\omega(x)$ such that $U - A \in \mathcal{I}$ and $V - B \in \mathcal{I}$. Let $G = U \cap V \in \tau_\omega(x)$. We have $G - A \in \mathcal{I}$ and $G - B \in \mathcal{I}$ by heredity. Thus, $G - (A \cap B) = (G - A) \cup (G - B) \in \mathcal{I}$ by additivity, and hence, $x \in \Psi_\omega(A \cap B)$. We have shown $\Psi_\omega(A) \cap \Psi_\omega(B) \subseteq \Psi_\omega(A \cap B)$ and the proof is complete.

(4) If $U \in \tau^*_\omega$, then $X - U$ is τ^*_ω-closed. This implies $(X - U)_\omega \subseteq X - U$. Hence, $U \subseteq X - (X - U)_\omega = \Psi_\omega(U)$.

(5) This follows from (4).

(6) This follows from the facts:

(1) $\Psi_\omega(A) = X - (X - A)_\omega$.
(2) $\Psi_\omega(\Psi_\omega(A)) = X - [X - (X - (X - A)_\omega)]_\omega = X - ((X - A)_\omega)_\omega$.

(7) By Corollary 1.4, we obtain that $(X - A)_\omega = X_\omega$, if $A \in \mathcal{I}$.

(8) If $x \in A \cap \Psi_\omega(A)$, then $x \in A$ and there exists $U_x \in \tau_\omega(x)$ such that $U_x - A \in \mathcal{I}$. Then by Theorem 1.7, $U_x - (U_x - A)$ is a τ^*_ω-open neighborhood

of x and $x \in Int^*_\omega(A)$. On the other hand, if $x \in Int^*_\omega(A)$, there exists a basic τ^*_ω-open neighborhood $V_x - I$ of x, where $V_x \in \tau_\omega$ and $I \in \mathcal{I}$, such that $x \in V_x - I \subseteq A$. This implies $V_x - A \subseteq I$. Hence, $V_x - A \in \mathcal{I}$. Therefore, $x \in A \cap \Psi_\omega(A)$.

(9) This follows from Corollary 1.4 and $\Psi_\omega(A-I) = X-[X-(A-I)]_\omega = X - [(X - A) \cup I]_\omega = X - (X - A)_\omega = \Psi_\omega(A)$.

(10) This follows from Corollary 1.4 and $\Psi_\omega(A \cup I) = X-[X-(A\cup I)]_\omega = X - [(X - A) - I]_\omega = X - (X - A)_\omega = \Psi_\omega(A)$.

(11) Assume $(A - B) \cup (B - A) \in \mathcal{I}$. Let $A - B = I$ and $B - A = J$. Observe that $I, J \in \mathcal{I}$ by heredity. Also, observe that $B = (A - I) \cup J$. Thus, $\Psi_\omega(A) = \Psi_\omega(A - I) = \Psi[(A - I) \cup J] = \Psi_\omega(B)$ by (9) and (10). □

Corollary 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $U \subseteq \Psi_\omega(U)$ for every ω-open set $U \in \tau$.*

Proof. We know that $\Psi_\omega(U) = X - (X - U)_\omega$. Now $(X - U)_\omega \subseteq Cl_\omega(X - U) = X - U$, since $X - U$ is ω-closed. Therefore, $U = X - (X - U) \subseteq X - (X - U)_\omega = \Psi_\omega(U)$. □

Theorem 2.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following properties hold:*

(1) $\Psi_\omega(A) = \cup\{U \in \tau_\omega : U - A \in \mathcal{I}\}$.
(2) $\Psi_\omega(A) \supseteq \cup\{U \in \tau_\omega : (U - A) \cup (A - U) \in \mathcal{I}\}$.

Proof.
(1) This follows immediately from the definition of Ψ_ω-operator.
(2) Since $\mathcal{I}$ is hereditary, it is obvious that $\cup\{U \in \tau_\omega : (U-A)\cup(A-U) \in \mathcal{I}\} \subseteq \cup\{U \in \tau_\omega : U - A \in \mathcal{I}\} = \Psi_\omega(A)$ for every $A \subseteq X$. □

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\sigma = \{A \subseteq X : A \subseteq \Psi_\omega(A)\}$. Then σ is a topology for X and $\sigma = \tau^*_\omega$.*

Proof. Let $\sigma = \{A \subseteq X : A \subseteq \Psi_\omega(A)\}$. First, we show that σ is a topology. Observe that $\phi \subseteq \Psi_\omega(\phi)$ and $X \subseteq \Psi_\omega(X) = X$, and thus, ϕ and $X \in \sigma$. Now if $A, B \in \sigma$, then $A \cap B \subseteq \Psi_\omega(A) \cap \Psi_\omega(B) = \Psi_\omega(A \cap B)$ which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \Psi_\omega(A_\alpha) \subseteq \Psi_\omega(\cup A_\alpha)$ for every α and hence, $\cup A_\alpha \subseteq \Psi_\omega(\cup A_\alpha)$. This shows that σ is a topology. Now if $U \in \tau^*_\omega$ and $x \in U$, then by Theorem 1.7, there exist $V \in \tau_\omega(x)$ and $I \in \mathcal{I}$ such that $x \in V - I \subseteq U$. Clearly $V - U \subseteq I$ so that $V - U \in \mathcal{I}$ by heredity and hence, $x \in \Psi_\omega(U)$. Thus, $U \subseteq \Psi_\omega(U)$ and we have shown $\tau^*_\omega \subseteq \sigma$. Now let $A \in \sigma$. Then we have $A \subseteq \Psi_\omega(A)$, that is, $A \subseteq X - (X - A)_\omega$ and $(X - A)_\omega \subseteq X - A$. This shows that $X - A$ is τ^*_ω-closed and hence, $A \in \tau^*_\omega$. Thus, $\sigma \subseteq \tau^*_\omega$ and hence, $\sigma = \tau^*_\omega$. □

3. Some Properties in ω-Compatible Spaces

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\tau \sim_\omega \mathcal{I}$ if and only if $\Psi_\omega(A) - A \in \mathcal{I}$ for every $A \subseteq X$.*

Proof.

Necessity. Assume $\tau \sim_\omega \mathcal{I}$ and let $A \subseteq X$. Observe that $x \in \Psi_\omega(A) - A \in \mathcal{I}$ if and only if $x \notin A$ and $x \notin (X - A)_\omega$ if and only if $x \notin A$ and there exists $U_x \in \tau_\omega(x)$ such that $U_x - A \in \mathcal{I}$ if and only if there exists $U_x \in \tau_\omega(x)$ such that $x \in U_x - A \in \mathcal{I}$. Now, for each $x \in \Psi_\omega(A) - A$ and $U_x \in \tau_\omega(x)$, $U_x \cap (\Psi_\omega(A) - A) \in \mathcal{I}$ by heredity and hence, $\Psi_\omega(A) - A \in \mathcal{I}$, by assumption that $\tau \sim_\omega \mathcal{I}$.

Sufficiency. Let $A \subseteq X$ and assume that for each $x \in A$, there exists $U_x \in \tau_\omega(x)$ such that $U_x \cap A \in \mathcal{I}$. Observe that $\Psi_\omega(X - A) - (X - A) = \{x :$ there exists $U_x \in \tau_\omega(x)$ such that $x \in U_x \cap A \in \mathcal{I}\}$. Thus, we have $A \subseteq \Psi_\omega(X - A) - (X - A) \in \mathcal{I}$ and hence, $A \in \mathcal{I}$ by heredity of $\mathcal{I}$. □

Lemma 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $\tau \sim_\omega \mathcal{I}$ and $A \subseteq X$. Then A is a τ^*_ω-closed if and only if $A = B \cup I$ such that B is ω-closed and $I \in \mathcal{I}$.*

Proof. If A is a τ^*_ω-closed set, then $A_\omega \subseteq A$. This implies that $A = A \cup A_\omega = (A - A_\omega) \cup A_\omega$. Then from Lemma 1.2, A_ω is an ω-closed set and from Lemma 1.11, $A - A_\omega \in \mathcal{I}$. Conversely, if $A = B \cup I$ such that B is an ω-closed set and $I \in \mathcal{I}$, then, by Corollary 1.4, we get that $A_\omega = (B \cup I)\omega = B_\omega \cup I_\omega = B_\omega \subseteq Cl_\omega(B) = B \subseteq A$. This implies that A is a τ^*_ω-closed. □

Corollary 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $\tau \sim_\omega \mathcal{I}$. Then $\beta(\mathcal{I}, \tau)$ is a topology on X and hence, $\beta(\mathcal{I}, \tau) = \tau^*_\omega$.*

Proof. Let $A \in \tau^*_\omega$. Then by Lemma 3.2, $X - A = F \cup I$, where F is ω-closed and $I \in \mathcal{I}$. Then $A = X - (F \cup I) = (X - F) \cap (X - I) = (X - F) - I = V - I$, where $V = X - F \in \tau_\omega$. Thus, every τ_ω-open set is of the form $V - I$, where $V \in \tau_\omega$ and $I \in \mathcal{I}$. The result follows by Theorem 1.7. □

Proposition 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\omega \mathcal{I}$ and $A \subseteq X$. If N is a nonempty ω-open subset of $A_\omega \cap \Psi_\omega(A)$, then $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof. If $N \subseteq A_\omega \cap \Psi_\omega(A)$, then $N - A \subseteq \Psi_\omega(A) - A \in \mathcal{I}$ by Theorem 3.1. Hence, $N - A \in \mathcal{I}$ by heredity. Since $N \in \tau_\omega - \{\phi\}$ and $N \subseteq A_\omega$, we have $N \cap A \notin \mathcal{I}$, by the definition of A_ω. □

As a consequence of the Theorem 3.2, we have the following.

Corollary 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\omega \mathcal{I}$. Then $\Psi_\omega(\Psi_\omega(A)) = \Psi_\omega(A)$ for every $A \subseteq X$.*

Proof. $\Psi_\omega(A) \subseteq \Psi_\omega(\Psi_\omega(A))$ follows from Theorem 2.2 (5). Since $\tau \sim_\omega \mathcal{I}$, it follows, from Theorem 3.1, that $\Psi_\omega(A) \subseteq A \cup I$ for some $I \in \mathcal{I}$ and hence, $\Psi_\omega(\Psi_\omega(A)) = \Psi_\omega(A)$ by Theorem 2.2 (10). □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\omega \mathcal{I}$. Then $\Psi_\omega(A) = \cup\{\Psi_\omega(U) : U \in \tau_\omega, \Psi_\omega(U) - A \in \mathcal{I}\}$.*

Proof. Let $\Phi(A) = \cup\{\Psi_\omega(U) : U \in \tau_\omega, \Psi_\omega(U) - A \in \mathcal{I}\}$. Clearly, $\Phi(A) \subseteq \Psi_\omega(A)$. Now let $x \in \Psi_\omega(A)$. Then there exists $U \in \tau_\omega(x)$ such that $U - A \in \mathcal{I}$. By Corollary 2.3, $U \subseteq \Psi_\omega(U)$ and $\Psi_\omega(U) - A \subseteq [\Psi_\omega(U) - U] \cup [U - A]$. By Theorem 3.1, $\Psi_\omega(U) - U \in \mathcal{I}$ and hence, $\Psi_\omega(U) - A \in \mathcal{I}$. Hence, $x \in \Phi(A)$ and $\Phi(A) \supseteq \Psi_\omega(A)$. Consequently, we obtain $\Phi(A) = \Psi_\omega(A)$. □

In [8], Newcomb defines $A = B \pmod{\mathcal{I}}$, if $(A - B) \cup (B - A) \in \mathcal{I}$ and observes that mod is an equivalence relation. By Theorem 2.2 (11), we have that if $A = B \pmod{\mathcal{I}}$, then $\Psi_\omega(A) = \Psi_\omega(B)$.

Definition 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called a Baire set with respect to τ and $\mathcal{I}$, denoted $A \in \mathcal{W}_r(X, \tau, \mathcal{I})$, if there exists an ω-open set $U \in \tau$ such that $A = U \pmod{\mathcal{I}}$.*

Lemma 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\omega \mathcal{I}$. If U, $V \in \tau_\omega$ and $\Psi_\omega(U) = \Psi_\omega(V)$, then $U = V \pmod{\mathcal{I}}$.*

Proof. Since $U \in \tau_\omega$, we have $U \subseteq \Psi_\omega(U)$ and hence, $U - V \subseteq \Psi_\omega(U) - V = \Psi_\omega(V) - V \in \mathcal{I}$ by Theorem 3.1. Similarly, $V - U \in \mathcal{I}$. Now $(U - V) \cup (V - U) \in \mathcal{I}$ by additivity. Hence, $U = V \pmod{\mathcal{I}}$. □

Theorem 3.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\tau \sim_\omega \mathcal{I}$. If A, $B \in \mathcal{W}_r(X, \tau, \mathcal{I})$, and $\Psi_\omega(A) = \Psi_\omega(B)$, then $A = B \pmod{\mathcal{I}}$.*

Proof. Let $U, V \in \tau_\omega$ such that $A = U \pmod{\mathcal{I}}$ and $B = V \pmod{\mathcal{I}}$. Now $\Psi_\omega(A) = \Psi_\omega(U)$ and $\Psi_\omega(B) = \Psi_\omega(V)$ by Theorem 2.2(11). Since $\Psi_\omega(A) = \Psi_\omega(U)$ implies that $\Psi_\omega(U) = \Psi_\omega(V)$, we have that $U = V \pmod{\mathcal{I}}$ by Lemma 3.8. Hence, $A = B \pmod{\mathcal{I}}$ by transitivity. □

4. ω-Codense in Ideal Topological Spaces

Proposition 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space.*

(1) *If $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then there exists $A \in \tau - \{\phi\}$ such that $B = A \pmod{\mathcal{I}}$.*
(2) *If $\mathcal{I}$ is ω-codense, then $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ if and only if there exists $A \in \tau - \{\phi\}$ such that $B = A \pmod{\mathcal{I}}$.*

Proof.
(1) Assume $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$. Then $B \in \mathcal{W}_r(X, \tau, \mathcal{I})$. Now if there

does not exist $A \in \tau - \{\phi\}$ such that $B = A \pmod{\mathcal{I}}$, we have $B = \phi \pmod{\mathcal{I}}$. This implies that $B \in \mathcal{I}$, which is a contradiction.

(2) Assume there exists $A \in \tau - \{\phi\}$ such that $B = A \pmod{\mathcal{I}}$. Then $A = (B - J) \cup I$, where $J = B - A, I = A - B \in \mathcal{I}$. If $B \in \mathcal{I}$, then $A \in \mathcal{I}$ by heredity and additivity, which contradicts that $\mathcal{I}$ is ω-codense. □

Proposition 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\mathcal{I}$ is ω-codense. If $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, then $\Psi_\omega(B) \cap Int_\omega(B_\omega) \neq \phi$.*

Proof. Assume $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$. Then by Proposition 4.1 (1), there exists $A \in \tau - \{\phi\}$ such that $B = A \pmod{\mathcal{I}}$. This implies that $\phi \neq A \subseteq A_\omega = ((B - J) \cup I)_\omega = B_\omega$, where $J = B - A$ and $I = A - B \in \mathcal{I}$, by Theorem 1.3 and Corollary 1.4. Also, $\phi \neq A \subseteq \Psi_\omega(A) = \Psi_\omega(B)$ by Theorem 2.2 (11), so that $A \subseteq \Psi_\omega(B) \cap Int_\omega(B_\omega)$. □

Given an ideal topological space $(X, \tau, \mathcal{I})$, let $\mathcal{U}(X, \tau, \mathcal{I})$ denote $\{A \subseteq X :$ there exists $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A\}$.

Proposition 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\mathcal{I}$ is ω-codense. The following properties are equivalent:*

(1) $A \in \mathcal{U}(X, \tau, \mathcal{I})$;
(2) $\Psi_\omega(A) \cap Int_\omega(A_\omega) \neq \phi$;
(3) $\Psi_\omega(A) \cap A_\omega \neq \phi$;
(4) $\Psi_\omega(A) \neq \phi$;
(5) $Int^*_\omega(A) \neq \phi$;
(6) *There exists $N \in \tau_\omega - \{\phi\}$ such that $N - A \in \mathcal{I}$ and $N \cap A \notin \mathcal{I}$.*

Proof.

(1) $\Rightarrow$ (2): Let $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$ such that $B \subseteq A$. Then $Int_\omega(B_\omega) \subseteq Int_\omega(A_\omega)$ and $\Psi_\omega(B) \subseteq \Psi_\omega(A)$ and hence, $Int_\omega(B_\omega) \cap \Psi_\omega(B) \subseteq Int_\omega(A_\omega) \cap \Psi_\omega(A)$. By Proposition 4.2, we have $\Psi_\omega(A) \cap Int_\omega(A_\omega) \neq \phi$.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): The proof is obvious.

(4) $\Rightarrow$ (5): If $\Psi_\omega(A) \neq \phi$, then there exists $U \in \tau_\omega - \{\phi\}$ such that $U - A \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - A) \cup (U \cap A)$, we have $U \cap A \notin \mathcal{I}$. By Theorem 2.2, $\phi \neq (U \cap A) \subseteq \Psi_\omega(U) \cap A = \Psi_\omega((U - A) \cup (U \cap A)) \cap A = \Psi_\omega(U \cap A) \cap A \subseteq \Psi_\omega(A) \cap A = Int^*_\omega(A)$. Hence, $Int^*_\omega(A) \neq \phi$.

(5) $\Rightarrow$ (6): If $Int^*_\omega(A) \neq \phi$, then by Theorem 1.7 there exists $N \in \tau_\omega - \{\phi\}$ and $I \in \mathcal{I}$ such that $\phi \neq N - I \subseteq A$. We have $N - A \in \mathcal{I}$, $N = (N - A) \cup (N \cap A)$ and $N \notin \mathcal{I}$. This implies that $N \cap A \notin \mathcal{I}$.

(6) $\Rightarrow$ (1): Let $B = N \cap A \notin \mathcal{I}$ with $N \in \tau_\omega - \{\phi\}$ and $N - A \in \mathcal{I}$. Then $B \in \mathcal{W}_r(X, \tau, \mathcal{I}) - \mathcal{I}$, since $B \notin \mathcal{I}$ and $(B - N) \cup (N - B) = N - A \in \mathcal{I}$. □

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\mathcal{I}$ is ω-codense. Then for $A \subseteq X$, $\Psi_\omega(A) \subseteq A_\omega$.*

Proof. Suppose $x \in \Psi_\omega(A)$ and $x \notin A_\omega$. Then there exists a nonempty neighborhood $U_x \in \tau_\omega(x)$ such that $U_x \cap A \in \mathcal{I}$. Since $x \in \Psi_\omega(A)$, by Theorem 2.4, $x \in \cup\{U \in \tau_\omega : U - A \in \mathcal{I}\}$ and there exists $V \in \tau_\omega$ such that $x \in V$ and $V - A \in \mathcal{I}$. Now we have $U_x \cap V \in \tau_\omega(x)$, $U_x \cap V \cap A \in \mathcal{I}$, and $(U_x \cap V) - A \in \mathcal{I}$ by heredity. Hence, by finite additivity, we have $(U_x \cap V \cap A) \cup (U_x \cap V - A) = (U_x \cap V) \in \mathcal{I}$. Since $(U_x \cap V) \in \tau_\omega(x)$, this is contrary to $\mathcal{I}$ is ω-codense. Therefore, $x \in A_\omega$. This implies that $\Psi_\omega(A) \subseteq A_\omega$. □

Corollary 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $\mathcal{I}$ is ω-codense. Then for $A \subseteq X$, $\Psi_\omega(A) \subseteq Cl_\omega(A_\omega)$.*

Theorem 4.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:*

(1) *$\mathcal{I}$ is ω-codense;*
(2) *$\Psi_\omega(\phi) = \phi$;*
(3) *If $A \subseteq X$ is ω-closed, then $\Psi_\omega(A) - A = \phi$;*
(4) *If $I \in \mathcal{I}$, then $\Psi_\omega(I) = \phi$.*

Proof.
(1) $\Rightarrow$ (2): Since $\mathcal{I}$ is ω-codense, by Theorem 2.4, we have $\Psi_\omega(\phi) = \cup\{U \in \tau_\omega : U \in \mathcal{I}\} = \phi$.

(2) $\Rightarrow$ (3): Suppose $x \in \Psi_\omega(A) - A$. Then there exists $U_x \in \tau_\omega(x)$ such that $x \in U_x - A \in \mathcal{I}$ and $U_x - A \in \tau_\omega$. But $U_x - A \in \{U \in \tau_\omega : U \in \mathcal{I}\} = \Psi_\omega(\phi)$, which implies that $\Psi_\omega(\phi) = \phi$. Hence, $\Psi_\omega(A) - A = \phi$.

(3) $\Rightarrow$ (4): Let $I \in \mathcal{I}$. Since ϕ is ω-closed, $\Psi_\omega(I) = \Psi_\omega(I \cup \phi) = \Psi_\omega(\phi) = \phi$.

(4) $\Rightarrow$ (1): Suppose $A \in \tau_\omega \cap \mathcal{I}$, then $A \in \mathcal{I}$ and by (4), $\Psi_\omega(A) = \phi$. Since $A \in \tau_\omega$, by Corollary 2.3, we have $A \subseteq \Psi_\omega(A) = \phi$. Hence, $\mathcal{I}$ is ω-codense. □

Theorem 4.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\mathcal{I}$ is ω-codense if and only if $[\Psi_\omega(A)]_\omega = Cl_\omega[\Psi_\omega(A)]$ for every $A \subseteq X$.*

Proof. Let $\mathcal{I}$ be ω-codense. It is obvious that $[\Psi_\omega(A)]_\omega \subseteq Cl_\omega[\Psi_\omega(A)]$. For the reverse inclusion, let $x \in Cl_\omega[\Psi_\omega(A)]$. Then for every ω-open sets U_x containing x, $U_x \cap \Psi_\omega(A) \neq \phi$ implies that $U_x \cap \Psi_\omega(A) \notin \mathcal{I}$, since $\mathcal{I}$ is ω-codense. Hence, $x \in [\Psi_\omega(A)]_\omega$. Hence, $[\Psi_\omega(A)]_\omega = Cl_\omega[\Psi_\omega(A)]$. Conversely, suppose that $[\Psi_\omega(A)]_\omega = Cl_\omega[\Psi_\omega(A)]$, for every $A \subseteq X$. Then for $X \subseteq X$, $[\Psi_\omega(X)]_\omega = Cl_\omega[\Psi_\omega(X)]$. Hence, $[X - (X - X)_\omega]_\omega = Cl_\omega[X - (X - X)_\omega]$, implies that $X_\omega = Cl_\omega(X) = X$. Hence, $\mathcal{I}$ is ω-codense. □

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $\tau \sim_\omega \mathcal{I}$ and $\mathcal{I}$ is ω-codense. Then*

(1) (X, τ_ω) *is Hausdorff or Urysohn if and only if* (X, τ^*_ω) *is respectively so.*
(2) *If* (X, τ^*_ω) *is regular then* $\tau_\omega = \tau^*_\omega$.
(3) (X, τ_ω) *is connected if and only if* (X, τ^*_ω) *is connected.*

Proof.
(1) Let (X, τ^*_ω) be Hausdorff and x and y be any two distinct points of X. Then there exists disjoint τ^*_ω-open sets G and H containing x and y, respectively. Then by Corollary 3.3, $G = U - I_1$ and $H = V - I_2$, where U, $V \in \tau_\omega$ and $I_1, I_2 \in \mathcal{I}$. Since U and V are ω-open sets containing x and y, respectively, it remains to show that $U \cap V = \phi$. Now $G \cap H = [U - I_1] \cap [V - I_2] = [U \cap V] - [I_1 \cup I_2] = \phi$. Then $U \cap V \subseteq I_1 \cup I_2$ and hence, $[U \cap V]_\omega \subseteq [I_1 \cup I_2]_\omega = [I_1]_\omega \cup [I_2]_\omega = \phi$, by Lemma 1.2 and Theorem 1.3. Since $\mathcal{I}$ is ω-codense, we have, by Lemma 1.12, that $U \cap V \subseteq [U \cap V]_\omega = \phi$, so that $U \cap V = \phi$. The converse is trivial.

Next, let (X, τ^*_ω) be Urysohn and x and y be two distinct points of X. Then there exists τ^*_ω-open sets G and H containing x and y, respectively and $Cl^*_\omega(G) \cap Cl^*_\omega(H) = \phi$, where by Corollary 3.3, we can take $G = U - I_1$ and $H = V - I_2$, where U, $V \in \tau_\omega$ and $I_1, I_2 \in \mathcal{I}$. Then $x \in U$, $y \in V$, and by Theorem 1.13, $Cl_\omega(U) \cap Cl_\omega(V) = \phi$. Hence, (X, τ_ω) is Urysohn.

Conversely, if (X, τ_ω) is Urysohn, then for $x, y \in X$ with $x \neq y$, there exist $U, V \in \tau_\omega$ such that $Cl_\omega(U) \cap Cl_\omega(V) = \phi$. Then U and V are τ^*_ω-open and, by Theorem 1.13, $Cl_\omega(U) = Cl^*_\omega(U)$ and $Cl_\omega(V) = Cl^*_\omega(V)$. Hence, (X, τ^*_ω) is Urysohn.

(2) For any $A \subseteq X$, we clearly have $Cl^*_\omega(A) \subseteq Cl_\omega(A)$. Let $x \notin Cl^*_\omega(A)$. Then for some τ^*_ω-open neighborhood of G of x, we have $G \cap A = \phi$. By regularity of (X, τ^*_ω), there exists $H \in \tau^*_\omega$ with $H = U - I$, where $U \in \tau_\omega$ and $I \in \mathcal{I}$, such that $x \in H \subseteq Cl^*_\omega(H) \subseteq G$. Now, $U \cap A \subseteq Cl_\omega(U) \cap A = Cl^*_\omega(H) \cap A \subseteq G \cap A = \phi$ by Theorem 1.13, and hence, $U \cap A = \phi$, where $x \in U \in \tau_\omega$ and $x \notin Cl_\omega(A)$. Hence, $Cl^*_\omega(A) = Cl_\omega(A)$ for each $A \subseteq X$ and $\tau_\omega = \tau^*_\omega$.

(3) If (X, τ^*_ω) is connected, then so is (X, τ_ω). Suppose (X, τ^*_ω) is not connected. Then there exists a nonempty τ^*_ω-clopen set $A \neq X$ and $X = A \cup (X - A)$ such that $X = X_\omega = [A \cup (X - A)]_\omega = A_\omega \cup (X - A)_\omega$. Now A and $X - A$ are τ^*_ω-closed, $A_\omega \cup (X - A)_\omega \subseteq A \cup (X - A)$, and hence, $A_\omega \cup (X - A)_\omega = \phi$. Again as A is τ^*_ω-open, by Theorem 1.13, $A_\omega = Cl_\omega(A) \neq \phi$. Similarly, $(X - A)_\omega = Cl_\omega(X - A) \neq \phi$. Thus, $X = Cl_\omega(A) \cup Cl_\omega(X - A)$ and $Cl_\omega(A) \cap Cl_\omega(X - A) = \phi$ and $Cl_\omega(A) \neq \phi \neq Cl_\omega(X - A)$. Hence, (X, τ_ω) is not connected. □

We recall that a topological space X is called quasi H-closed (QHC, for short) [9] if every open cover of X has a finite subcollection, the union of its closures cover of X.

Theorem 4.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $\mathcal{I}$ is ω-codense. Then (X, τ_ω) is QHC if and only if (X, τ^*_ω) is QHC.*

Proof. Let (X, τ_ω) be QHC and let $\mathcal{U} = \{U_\alpha - I_\alpha : U_\alpha \in \tau_\omega,\ I_\alpha \in \mathcal{I},\ \alpha \in \Lambda\}$ be a τ^*_ω-basic open cover of X. Then $\{U_\alpha : \alpha \in \Lambda\}$ is a τ_ω-open cover of X. By quasi H-closedness of (X, τ_ω), there exist finitely many α, say, $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n \in \Lambda$, such that $X = \cup_{i=1}^{n} Cl_\omega(U_{\alpha_i})$. We are to show that $X = \cup_{i=1}^{n} Cl^*_\omega(U_{\alpha_i} - I_{\alpha_i})$. Suppose $x \in X = \cup_{i=1}^{n} Cl_\omega(U_{\alpha_i})$ such that $x \notin \cup_{i=1}^{n} Cl^*_\omega(U_{\alpha_i} - I_{\alpha_i})$. Then $x \notin Cl^*_\omega(U_{\alpha_i} - I_{\alpha_i})$ for each $i = 1, 2, \ldots, n$, while for some $\alpha_k \in \{\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda\}$, $x \in Cl_\omega(U_{\alpha_k})$. Since $x \notin Cl^*_\omega(U_{\alpha_i} - I_{\alpha_i})$, we get $G_i = V_i - I_i$ with $V_i \in \tau_\omega$ and $I_i \in \mathcal{I}$ such that $x \in G_i$ and $G_i \cap [U_{\alpha_i} - I_{\alpha_i}] = \phi$ for $i = 1, 2, \ldots, n$. Now $x \in G = G_1 \cap G_2 \cap \cdots \cap G_n = [V_1 \cap V_2 \cap \cdots \cap V_n] - [I_1 \cup I_2 \cup \cdots \cup I_n] \in \tau^*_\omega$. This implies that $G \cap [U_{\alpha_k} - I_{\alpha_k}] = \phi$ and $U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n] \neq \phi$, and so, $U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n] \notin \mathcal{I}$. To arrive at a contradiction, we only show that $U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n] \subseteq I_{\alpha_k} \cup [I_1 \cup I_2 \cup \cdots \cup I_n] \in \mathcal{I}$. Let $z \in U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n]$. Then, as $\phi = G \cap [U_{\alpha_k} - I_{\alpha_k}] = [(V_1 \cap V_2 \cap \cdots \cap V_n) - (I_1 \cup I_2 \cup \cdots \cup I_n)] \cap [U_{\alpha_k} - I_{\alpha_k}]$, we have $z \in (I_1 \cup I_2 \cup \cdots \cup I_n)$ or $z \in I_{\alpha_k}$. Hence, $z \in (I_1 \cup I_2 \cup \cdots \cup I_n) \cup I_{\alpha_k}$. This completes the proof. □

Definition 4.10. *A subset A in an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_\omega$-dense if $A_\omega = X$.*

Definition 4.11. *A topological space (X, τ) is said to be ω-hyperconnected if and only if every pair of nonempty ω-open sets U and V has a nonempty intersection, i.e., $U \cap V \neq \emptyset$.*

Proposition 4.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following properties are equivalent:*

(1) *Every nonempty ω-open set is $\mathcal{I}_\omega$-dense;*
(2) *(X, τ) is ω-hyperconnected and $\mathcal{I}$ is ω-codense.*

Proof.
(1) ⇒ (2): Clearly every nonempty ω-open set which is $\mathcal{I}_\omega$-dense is ω-hyperconnected. Let A be ω-open, nonempty and a member of the ideal. By (1), $A_\omega = X$. On the other hand, since $A \in \mathcal{I}$, $A_\omega = \phi$. Hence, $X = \phi$. By contradiction, $\mathcal{I}$ is ω-codense.

(2) ⇒ (1): Let $\phi \neq A \in \tau_\omega$. Let $x \in X$. Due to the ω-hyperconnectedness of (X, τ), every ω-open neighborhood V of x meets A. Moreover, $A \cap V$ is an ω-open non-ideal set, since $\mathcal{I}$ is ω-codense. Thus, $x \in A_\omega$. This shows that $A_\omega = X$ and A is $\mathcal{I}_\omega$-dense. □

Definition 4.13. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-resolvable (resp. $\mathcal{I}_\omega$-resolvable) if X has two disjoint $\mathcal{I}$-dense (resp. $\mathcal{I}_\omega$-dense) subsets.*

Lemma 4.14. *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}_\omega$-resolvable, then $\mathcal{I}$ is ω-codense.*

Proof. If $X = A \cup B$, where A and B are disjoint $\mathcal{I}_\omega$-dense, then $A_\omega = X$ and $B_\omega = X$. Therefore, $\tau_\omega(X) \cap A \notin \mathcal{I}$ and $\tau_\omega(X) \cap B \notin \mathcal{I}$. Hence, $\tau_\omega \cap \mathcal{I} = \phi$, and $\mathcal{I}$ is ω-codense. □

Proposition 4.15. *Every $\mathcal{I}_\omega$-resolvable ideal topological space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable.*

Proof. If $X = A \cup B$, where A and B are disjoint $\mathcal{I}_\omega$-dense, then $A_\omega = X$ and $B_\omega = X$. Therefore, $X = A_\omega \subseteq A^*$ and $X = B_\omega \subseteq B^*$, and we get $X = A^*$ and $X = B^*$. Hence, $X = A \cup B$, where A and B are disjoint $\mathcal{I}$-dense and $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable. □

The collection of all $\mathcal{I}_\omega$-dense in $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}_\omega D(X, \tau)$. The collection of all dense sets in (X, τ) is denoted by $D(X, \tau)$. Now we show that the collection of dense sets in a topological space (X, τ_ω^*) and the collection of $\mathcal{I}_\omega$-dense sets in ideal topological space $(X, \tau, \mathcal{I})$ are equal if $\mathcal{I}$ is ω-codense.

Theorem 4.16. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is ω-codense, then $\mathcal{I}_\omega D(X, \tau) = D(X, \tau_\omega^*)$.*

Proof. Let $D \in \mathcal{I}_\omega D(X, \tau)$. Then $Cl_\omega^*(D) = D \cup D_\omega = X$, i.e., $D \in D(X, \tau_\omega^*)$. Therefore, $\mathcal{I}_\omega D(X, \tau) \subseteq D(X, \tau_\omega^*)$.

Conversely, let $D \in D(X, \tau_\omega^*)$. Then $Cl_\omega^*(D) = D \cup D_\omega = X$. We prove that $D_\omega = X$. Let $x \in X$ such that $x \notin D_\omega$. Therefore, there exists $\phi \neq U \in \tau_\omega$ such that $U \cap D \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U \cap (X - D) \notin \mathcal{I}$, $U \cap (X - D) \neq \phi$. Let $x_0 \in U \cap (X - D)$. Then $x_0 \notin D$ and also $x_0 \notin D_\omega$. But $x_0 \in D_\omega$ implies that $U \cap D \notin \mathcal{I}$. This is contrary to $U \cap D \in \mathcal{I}$. Thus, $x_0 \notin D \cup D_\omega = Cl_\omega^*(D) = X$. This is a contradiction. Therefore, we obtain $D \in \mathcal{I}_\omega D(X, \tau)$. Therefore, $D(X, \tau_\omega^*) \subseteq \mathcal{I}_\omega D(X, \tau)$. Hence, $\mathcal{I}_\omega D(X, \tau) = D(X, \tau_\omega^*)$. □

Theorem 4.17. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for $x \in X$, $X - \{x\}$ is $\mathcal{I}_\omega$-dense if and only if $\Psi_\omega(\{x\}) = \phi$.*

Proof. The proof follows from the definition of $\mathcal{I}_\omega$-dense sets, since
$\Psi_\omega(\{x\}) = X - (X - \{x\})_\omega = \phi$ if and only if $X = (X - \{x\})_\omega$. □

Proposition 4.18. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space.*
$A \nsubseteq Cl_\omega[\Psi_\omega(A)]$ if and only if there exists $x \in A$ such that there is an ω-open set V_x of x for which $X - A$ is $\mathcal{I}_\omega$-dense in V_x.

Proof. Let $A \nsubseteq Cl_\omega[\Psi_\omega(A)]$. There exists $x \in X$ such that $x \in A$, but $x \notin Cl_\omega[\Psi_\omega(A)]$. Hence, there exists an ω-open set V_x of x such that $V_x \cap \Psi_\omega(A) = \phi$. This implies that $V_x \cap [X - (X - A)_\omega] = \phi$ and so $V_x \subseteq (X - A)_\omega$. Let U be any nonempty ω-open set in V_x. Since $V_x \subseteq (X - A)_\omega$,

$U \cap (X - A) \notin \mathcal{I}$. This implies that $X - A$ is $\mathcal{I}_\omega$-dense in V_x. The converse is obvious by reversing process. □

Proposition 4.19. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $\mathcal{I}$ is ω-codense. Then $\Psi_\omega(A) \neq \phi$ if and only if A contains a nonempty τ^*_ω-interior.*

Proof. Let $\Psi_\omega(A) \neq \phi$. By Theorem 2.4 (1), $\Psi_\omega(A) = \cup\{U \in \tau_\omega : U - A \in \mathcal{I}\}$ and there exists a nonempty set $U \in \tau_\omega$ such that $U - A \in \mathcal{I}$. Let $U - A = P$, where $P \in \mathcal{I}$. Now $U - P \subseteq A$. By Theorem 1.7, $U - P \in \tau^*_\omega$ and A contains a nonempty τ^*_ω-interior.

Conversely, suppose that A contains a nonempty τ^*_ω-interior. Hence, there exists $U \in \tau_\omega$ and $P \in \mathcal{I}$ such that $U - P \subseteq A$. So $U - A \subseteq P$. Let $H = U - A \subseteq P$. Then $H \in \mathcal{I}$. Hence, $\cup\{U \in \tau_\omega : U - A \in \mathcal{I}\} = \Psi_\omega(A) \neq \phi$. □

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