MAGNIFYING ELEMENTS IN A SEMIGROUP OF TRANSFORMATIONS WITH RESTRICTED RANGE

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Abstract. Let Y be a nonempty subset of a set X and let $T(X,Y)$ be the semigroup (under composition) of all functions $X \to X$ whose range is a subset of Y. We give necessary and sufficient conditions for elements in $T(X,Y)$ to be left and right magnifying.

1. Introduction and Preliminary

The notions of left and right magnifying elements of a semigroup were introduced by Ljapin (Chapter 3 in [5]). We recall that an element a of a semigroup S is called *left* [*right*] *magnifying* if there exists a proper subset M of S such that $S = aM$ [$S = Ma$]. For some properties of left and right magnifying elements in semigroups, see [2, 3, 4, 6, 7, 8]. In [2], Catino and Migliorini gave a necessary and sufficient condition for any semigroup to contain left magnifying elements and right magnifying elements. In [3], Gutan showed that every semigroup containing magnifying elements is factorizable. Let X be a nonempty set and let $T(X)$ denote the set of all transformations from X into itself, that is, $T(X) = \{f : X \to X \mid f \text{ is a function }\}$. It is well-known that $T(X)$ is a semigroup under composition (called the full transformation semigroup). It plays an important role in semigroup theory (it is known, for example, every semigroup is isomorphic to a subsemigroup of a suitable full transformation semigroup). In [6], Magill, Jr. studied left magnifying elements and right magnifying elements in transformation semigroups and applied to the linear transformation semigroups over a vector space V and the semigroup of all continuous selfmaps of a topological space X.

In this paper, we will write functions from the right, $(x)f$ rather than $f(x)$ and compose from left to right, $(x)(fg)$ rather than $(g \circ f)(x)$, for $f, g \in T(X)$ and $x \in X$. Let Y be a fixed nonempty subset of a set X. Let $T(X,Y) = \{f \in T(X) \mid ran\, f \subseteq Y\}$. Then $T(X,Y)$ is a subsemigroup of $T(X)$. Clearly, if $|Y| = 1$, then $T(X,Y)$ contains exactly one element. If

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$Y = X$, then $T(X, Y) = T(X)$. In [13], Symons described all the automorphisms of $T(X, Y)$. In [9], Nenthein, Youngkhong, and Kemprasit determined its regular elements. In [12], Sanwong, Singha, and Sullivan characterized all maximal and minimal congruences on $T(X, Y)$. In [11], Sanwong and Sommanee determined the largest regular subsemigroup of $T(X, Y)$ when $|Y| \neq 1$ and $Y \neq X$ and used that to describe Green's relations on $T(X, Y)$. In [10], Sanwong described Green's relations and ideals of some subsemigroup of $T(X, Y)$ and obtained from that all of its maximal regular subsemigroups when Y is a nonempty finite subset of X. In [1], Anantayasethi and Koppitz characterized the maximal regular subsemigroups of some subsemigroup of $T(X, Y)$.

Our aim in this paper is to give necessary and sufficient conditions for elements in $T(X, Y)$ to be left (respectively right) magnifying elements.

2. Right Magnifying Elements

Lemma 2.1. *If f is a right magnifying element in $T(X, Y)$, then f is onto.*

Proof. Assume f is a right magnifying element in $T(X, Y)$. Then there exists a proper subset M of $T(X, Y)$ such that $Mf = T(X, Y)$. Since $Y \subseteq X$, there exists an onto function g in $T(X, Y)$. Thus, there exists $h \in M$ such that $hf = g$. This implies f is onto. □

Lemma 2.2. *Let $f \in T(X, Y)$ be onto but not one-to-one.*

1. *If $(y)f^{-1} \cap Y = \emptyset$ for some $y \in Y$, then f is not right magnifying.*
2. *If $|(y)f^{-1} \cap Y| = 1$ for all $y \in Y$, then f is not right magnifying.*
3. *If $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$, then f is right magnifying.*

Proof. Let $f \in T(X, Y)$ be onto but not one-to-one.

(1) Let $y_0 \in Y$ be such that $(y_0)f^{-1} \cap Y = \emptyset$ and let $g \in T(X, Y)$ be such that $(x)g = y_0$ for all $x \in X$. Then there is no $h \in T(X, Y)$ such that $hf = g$. Therefore, f is not right magnifying.

(2) Assume $|(y)f^{-1} \cap Y| = 1$ for all $y \in Y$. Then $f|_Y$ is bijective. Assume f is right magnifying. Then there exists a proper subset M of $T(X, Y)$ such that $Mf = T(X, Y)$. Hence, $Mf = T(X, Y)f$. Since $f|_Y$ is bijective, $M = T(X, Y)$, a contradiction. Then f is not right magnifying.

(3) Assume $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$. Let $M = \{h \in T(X, Y) \mid h \text{ is not onto }\}$. Then $M \neq T(X, Y)$. Let g be any function in $T(X, Y)$. Since f is onto and $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, there exists for each $x \in X$, an element $y_x \in Y$ such that $(y_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $y_{x_1} = y_{x_2}$). Define $h \in T(X, Y)$ by $(x)h = y_x$ for all $x \in X$. We claim that h is not onto. Since $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$, there exist an element $y' \in Y$ and distinct

elements $y_1, y_2 \in Y$ such that $(y_1)f = (y_2)f = y'$. If $y' \notin \operatorname{ran} g$, we have $y_1, y_2 \notin \operatorname{ran} h$. If $y' \in ran\, g$, there is only one between y_1 and y_2 in $\operatorname{ran} h$. Then h is not onto. Hence, $h \in M$ and for all $x \in X$, we have

$$(x)hf = (y_x)f = (x)g.$$

Then $hf = g$, hence, $Mf = T(X, Y)$. Therefore, f is right magnifying. □

Theorem 2.3. *A function f in $T(X, Y)$ is right magnifying if and only if f is onto but not one-to-one and is such that $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$.*

Proof. Assume f is right magnifying. By Lemma 2.1, f is onto. Suppose f is one-to-one. Since f is right magnifying, there exists a proper subset M of $T(X, Y)$ such that $Mf = T(X, Y)$. This implies $Mf = T(X, Y)f$. Since f is one-to-one, $M = T(X, Y)$, this is a contradiction. Hence, f is not one-to-one. By Lemma 2.2, we have f is onto but not one-to-one such that $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$. Conversely, assume f is onto but not one-to-one and such that $(y)f^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)f^{-1} \cap Y| > 1$ for some $y \in Y$. By Lemma 2.2, f is right magnifying. □

Corollary 2.4. *Let $f \in T(X)$. Then f is right magnifying in $T(X)$ if and only if f is onto but not one-to-one.*

Proof. This follows directly from Theorem 2.3. □

3. Left Magnifying Elements

Lemma 3.1. *Suppose $|Y| < |X|$, then $T(X, Y)$ has no left magnifying element.*

Proof. If $|Y| = 1$, then $|T(X, Y)| = 1$, and so $T(X, Y)$ has no left magnifying element. Assume $|Y| > 1$. Let f be a left magnifying element in $T(X, Y)$. Then there exists a proper subset M of $T(X, Y)$ such that $fM = T(X, Y)$. Since $|Y| < |X|$, f is not one-to-one and so there exist $y \in Y$ and distinct elements $x_1, x_2 \in X$ such that $(x_1)f = (x_2)f = y$. Let $y' \in Y$ be such that $y' \neq y$ and define a function $g : X \to Y$ by

$$(x)g = \begin{cases} y & \text{if } x = x_1 \\ y' & \text{if } x \neq x_1. \end{cases}$$

Then there is no $h \in T(X, Y)$ such that $fh = g$, a contradiction. Hence, $T(X, Y)$ has no left magnifying element. □

Lemma 3.2. *Assume $|Y| = |X|$. If f is a left magnifying element in $T(X, Y)$, then f is one-to-one.*

Proof. Assume f is a left magnifying element in $T(X,Y)$. Then there exists a proper subset M of $T(X,Y)$ such that $fM = T(X,Y)$. Since $|X| = |Y|$, there exists a one-to-one function h in $T(X,Y)$. Therefore there exists $g \in M$ such that $fg = h$. This implies f is one-to-one. □

Lemma 3.3. *Assume $|Y| = |X|$ but $Y \neq X$. If f is one-to-one in $T(X,Y)$, then f is a left magnifying element in $T(X,Y)$.*

Proof. Assume $|Y| = |X|$, $Y \neq X$, and f is one-to-one. Let $y_0 \in Y$ and $M = \{h \in T(X,Y) \mid (x)h = y_0 \text{ for all } x \notin \mathit{ran}\, f\}$. We claim that $fM = T(X,Y)$. To see that, let $g \in T(X,Y)$ and define a function $h : X \to Y$ by

$$(x)h = \begin{cases} (x')g & \text{if } x \in \operatorname{ran} f \text{ and } (x')f = x, \\ y_0 & \text{if } x \notin \operatorname{ran} f. \end{cases}$$

Then $h \in M$ and for $x \in X$, we have

$$(x)fh = (x)g.$$

Hence, $fh = g$, and so $fM = T(X,Y)$. Since M is a proper subset of $T(X,Y)$, f is a left magnifying element in $T(X,Y)$. □

Theorem 3.4. *Assume $|X| = |Y|$ and $Y \neq X$. Then f is left magnifying of $T(X,Y)$ if and only if f is one-to-one.*

Proof. This follows from Lemma 3.2 and Lemma 3.3. □

Theorem 3.5. *A function f in $T(X)$ is a left magnifying element if and only if f is one-to-one but not onto.*

Proof. Assume f is one-to-one but not onto. Let $y_0 \in X$ and $M = \{h \in T(X) \mid (x)h = y_0 \text{ for all } x \notin \operatorname{ran} f\}$. We claim that $fM = T(X)$. Let $g \in T(X)$. Define a function $h : X \to X$ by

$$(x)h = \begin{cases} (x')g & \text{if } x \in \operatorname{ran} f \text{ and } (x')f = x, \\ y_0 & \text{if } x \notin \operatorname{ran} f. \end{cases}$$

Then $h \in M$ and for $x \in X$, we have

$$(x)fh = ((x)f)h = (x)g.$$

Hence, $fh = g$, and so $fM = T(X)$. Since M is a proper subset of $T(X)$, f is a left magnifying element in $T(X)$. Conversely, assume f is a left magnifying element in $T(X)$. By Lemma 3.2, f is one-to-one. Assume f is onto. Since f is bijective, its inverse function f^{-1} exists. Since f is a left magnifying element in $T(X)$, there exists a proper subset M of $T(X)$ such that $fM = T(X)$. We have $T(X) = f^{-1}T(X) = f^{-1}fM = M$, this is a contradiction. Hence, f is not onto. □

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References

[1] A. Anantayasethi and J. Koppitz, *On a semigroup of sets of transformations with restricted range*, Thai J. Math., **14** (2016), 667–676.

[2] F. Catino and F. Migliorini, *Magnifying elements in semigroups*, Semigroup Forum, **44** (1992), 314–319.

[3] M. Gutan, *Semigroups which contain magnifying elements are factorizable*, Commun. Algebra, **25** (1997), 3953–3963.

[4] M. Gutan, *Semigroups with strong and non strong magnifying elements*, Semigroup Forum, **53** (1996), 384–386.

[5] E. S. Ljapin, *Semigroups*, Transl. Math. Monographs, Vol. 3, Providence, RI, 1963.

[6] K. D. Magill, Jr., *Magnifying elements of transformation semigroups*, Semigroup Forum, **48** (1994), 119–126.

[7] F. Migliorini, *Some research on semigroups with magnifying elements*, Period. Math. Hung., **1** (1971), 279–286.

[8] F. Migliorini, *Magnifying elements and minimal subsemigroups in semigroups*, Period. Math. Hung., **5** (1974), 279–288.

[9] S. Nenthein, P. Youngkhong, and Y. Kemprasit, *Regular elements of some transformation semigroups*, Pure Math. Appl., **16** (2005), 307–314.

[10] J. Sanwong, *The regular part of a semigroup of transformations with restricted range*, Semigroup Forum, **83** (2011), 134–146.

[11] J. Sanwong and W. Sommanee, *Regularity and Green's relations on a semigroup of transformations with restricted range*, Int. J. Math. Math. Sci., **2008** (2008), 11 pp, Art. ID 794013.

[12] J. Sanwong, B. Singha, and R. P. Sullivan, *Maximal and minimal congruences on some semigroups*, Acta Math. Sin. (Engl. Ser.), **25** (2009), 455–466.

[13] J. S. V. Symons, *Some results concerning a transformation semigroup*, J. Aust. Math. Soc. (Ser. A), **19** (1975), 413–425.

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