# NEW TYPE OF SIMULTANEOUS REMOTAL SETS IN CERTAIN BANACH SPACES

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Abstract. In this paper, we introduce a new concept of simultaneous remotal sets and farthest points in Banach spaces and we present various characterizations of such points in certain Banach spaces.

## 1. Introduction

Let $X$ be a Banach space and $S$ be a closed bounded subset of $X$. For $x \in X$, set

$$\rho(x, S) = \sup_{s \in S} \|x - s\| \left( d(x, S) = \inf_{s \in S} \|x - s\| \right).$$

A point $s_0 \in S$ satisfying $\rho(x, S) = \|x - s_0\|$ $(d(x, S) = \|x - s_0\|)$ is called a farthest (nearest) point to $x$ from $S$. The set

$$F_S(x) = \{s \in S : \|s - x\| = \rho(x, S)\}$$
$$P_S(x) = \{s \in S : \|s - x\| = d(x, S)\}$$

is called the set of all farthest (nearest) points to $x$ from $S$. A bounded closed set $S$ is called remotal if for each $x \in X$, the set $F_S(x)$ is not empty [1, 2]. If a bounded set $W$ is given in $X$, one might like to approximate all elements of $W$ simultaneously by a single element of $S$. This type of problem rises when the function to be approximated is not known precisely, but it is known to belong to a set. Several mathematicians have studied this problem of simultaneous approximation in linear spaces, see [3, 4]. The problem of characterizing remotal sets in Banach spaces is an interesting problem. However, it is much more difficult than the problem of approximation. Furthermore, it has applications in approximation theory and geometry of Banach spaces. Some results regarding farthest points in Banach spaces are available in literature see for example, [5, 6, 7, 9].

There are many ways to approximate a set of points simultaneously by a point in a set $S$, see [3, 4]. In this paper we use the following definition.

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**Definition 1.1.** *Let $X$ be a real Banach space and $X^*$ be the dual space of $X$. For non-empty bounded subsets $W$ and $S$ of $X$, define $\rho(S,W) = \sup_{s\in S}\sup_{w\in W}\|s-w\|$. A point $s_0 \in S$ is called a simultaneous farthest point to $W$ from $S$ if $\sup_{w\in W}\|s_0-w\| = \rho(S,W)$. The set*

$$F_S(W) = \left\{ s \in S : \sup_{w\in W}\|s-w\| = \rho(S,W) \right\}$$

*is called the set of all simultaneous farthest points to $W$ from $S$.*

We remark that if $W$ is a singleton, then a simultaneous farthest point is precisely a farthest point.

It is easy to see that a compact set and remotality compact set $S$ with respect to a bounded set $W$ (i.e. any sequence $s_n$ in $S$ that satisfies $\sup_{w\in W}\|s_n-w\| \to \rho(S,W)$ is compact in $S$) admit a simultaneous farthest point.

For a non-empty subset $W$ of a Banach space $X$, the polar set $W^0$ of the set $W$ is defined to be $W^0 = \{f \in X^* : f(w) \le 0 \text{ for every } w \in W\}$, where $X^*$ is the dual space of $X$.

In this paper, we present various characterizations of simultaneous farthest points in certain Banach spaces in terms of the extremal points of the closed unit ball $B_{X^*}$ of $X^*$, where $X^*$ is the dual space of $X$.

Throughout this paper, $X^*$ is the dual of a Banach space $X$, $B_{X^*}$ is the unit ball of $X^*$ and $B(x_0,r) = \{x \in X : \|x-x_0\| \le r\}$. For any subset $W$ of $X$, we shall denote by $\operatorname{int}(W)$, $\operatorname{cl}(W)$, $\operatorname{clco}(W)$, and $\operatorname{bd}(W)$, the interior, the closure, the closed convex hull, and the boundary of $W$, respectively.

## 2. Main Results

In this section, we present various characterizations of simultaneous farthest points to a remotal set $W$ from a bounded set $S$ in certain Banach spaces in terms of the extremal points of the closed unit ball of $X^*$, where $X^*$ is the dual space of $X$. First we begin with the following proposition.

**Proposition 2.1** (Sphere reflection of sets)**.** *([7]) Let $X$ be a Banach space and $G$ be a closed bounded set in $X$. For $x \in (X-G)$, let $\rho(x,G) = r$ and $S(x,r) = \{y : \|x-y\| = r\}$. Then the map $\psi_x : G \to X$, $\psi_x(g) = 2x + 2r\frac{g-x}{\|g-x\|} - g$ has the following properties.*

1. *$\psi_x$ is continuous.*
2. *$\psi_x(G)$ is closed and bounded.*
3. *$d(S,G) = \inf\{\|s-g\| : s \in S \text{ and } g \in G\} = 0$.*
4. *$d(G,\psi_x(G)) = 0$.*
5. *$d(x,\psi_x(G)) = r$.*

(6) *$x$ has a farthest point in $G$ if and only if $x$ has a closest element in $\psi_x(G)$.*

**Definition 2.2** (The mirror reflection property)**.** *([7]) Let $X$ be a Banach space. We say that $X$ has the mirror reflection property if for any closed and bounded set $G \subset X$, and any $x \in (X - G)$, there exists a closed convex set $E \subset G$, such that $\rho(x, G) = \rho(x, E)$ and $\psi_x(E)$ is convex.*

It is known [7] that every finite dimensional normed space has the mirror reflection property.

Now we are able to prove one of the main results in this paper.

**Theorem 2.3.** *Let $X$ be a real Banach space which has the mirror reflection property and $W$ be a non-empty convex remotal subset of $X$. If $S$ is a non-empty bounded subset of $X$ such that $S \cap W = \phi$ and $s_0 \in S$, then the following are equivalent.*

(1) *$s_0 \in F_S(W)$.*

(2) *There exist $g \in X^*$, $w_0 \in F_W(s_0)$, and a subset $E \subset W$ with $\psi_{s_0}(E) \cap W \subset bd(B(s_0, r))$, where $r = \sup_{w \in W} \|s_0 - w\| = \|s_0 - w_0\|$ such that $\|g\| = 1$ and*

$$g \in (\psi_{s_0}(E) - w_0)^0, \ g \in (S - s_0)^0. \tag{2.1}$$

$$g(s_0 - w_0) = \sup_{s \in S} \sup_{w \in W} \|s - w\|. \tag{2.2}$$

*Proof.*

(1) $\Rightarrow$ (2).

Let $s_0 \in F_S(W)$. Then $\rho(S, W) = \sup_{w \in W} \|s_0 - w\|$. Since $W$ is remotal, there exists $w_0 \in W$ such that

$$\rho(s_0, W) = \sup_{w \in W} \|s_0 - w\| = \|s_0 - w_0\|.$$

Consequently, $w_0 \in F_W(s_0)$. Let

$$r = \sup_{w \in W} \|s_0 - w\| = \|s_0 - w_0\|.$$

Since $S \cap W = \phi$ and $W$ is closed, it follows that $r > 0$. Since $X$ has the mirror reflection property, there exists a closed convex set $E \subseteq W$ such that $\rho(s_0, W) = \rho(s_0, E)$ and $\psi_{s_0}(E)$ is convex. Furthermore, using Proposition 2.1, we obtain

$$\begin{aligned} d(s_0, \psi_{s_0}(E)) &= \inf_{e \in E} \|s_0 - \psi_{s_0}(e)\| = \sup_{e \in E} \|s_0 - e\| \\ &= \sup_{w \in W} \|s_0 - w\| = \rho(s_0, W) = \|s_0 - w_0\|. \end{aligned}$$

Now, we claim that $\psi_{s_0}(E) \cap W \subseteq \operatorname{bd}(B(s_0, r))$. Suppose that there is $z \in \psi_{s_0}(E) \cap W$. Then

$$\|z - s_0\| \leq \|w_0 - s_0\|.$$

But since $\|w_0 - s_0\| = \inf_{e \in E} \|s_0 - \psi_{s_0}(e)\|$, we have

$$\|z - s_0\| \geqslant \|w_0 - s_0\|.$$

Hence, $\|z - s_0\| = \|w_0 - s_0\| = r$. Thus, $z \in \operatorname{bd}(B(s_0, r))$. Note that $B(s_0, r)$ is a closed convex subset of $X$. Consequently, by the well-known Corollary of the Hahn-Banach Theorem, there exist $0 \neq f \in X^*$ and a real number $\lambda$ such that

$$(s_0 - \psi_{s_0}(e)) \geqslant \lambda \quad \text{for every} \quad e \in E$$

and

$$f(s_0 - y) \leq \lambda \quad \text{for every} \quad y \in B(s_0, r).$$

This implies that $\lambda = f(s_0 - w_0) \neq 0$. Let $g = \lambda^{-1} r f$. Then $g \in X^*$ and

$$g(s_0 - \psi_{s_0}(e)) \geqslant r \quad \text{for every} \quad e \in E,$$

and

$$g(s_0 - y) \leq r \quad \text{for every} \quad y \in B(s_0, r).$$

Also, we have

$$g(s_0 - w_0) = r = \|s_0 - w_0\|.$$

This implies that $\|g\| \geqslant 1$. We claim that $\|g\| = 1$. If not, then there exists $x \in X$ with $\|x\| = 1$ and $g(x) > 1$. Let $y_0 = s_0 - rx \in X$. It follows that $\|y_0 - s_0\| = r$. Hence, $y_0 \in B(s_0, r)$. But $g(s_0 - y_0) = rg(x) > r$. This contradicts the fact that $g(s_0 - y) \leq r$, for every $y \in B(s_0, r)$. Consequently, $\|g\| = 1$. Also, since $g(s_0 - \psi_{s_0}(e)) \geqslant r$ for every $e \in E$, we conclude that

$$g(\psi_{s_0}(e) - w_0) = g(\psi_{s_0}(e) - s_0) + g(s_0 - w_0) \leq -r + r = 0,$$

for every $e \in E$. Thus, $g \in (\psi_{s_0}(e) - w_0)^0$. Now for $s \in S$, the inequality

$$\begin{aligned} g(s - s_0) &= g(s - w_0) - g(s_0 - w_0) \\ &\leq \|g\| \|s - w_0\| - r \\ &\leq \sup_{s \in S} \sup_{w \in W} \|s - w\| - r \leq r - r = 0 \end{aligned}$$

implies that $g \in (S - s_0)^0$.

(2) $\Rightarrow$ (1).

Assume that (2) holds. Then there exists $w_0 \in W$ such that

$$\|s_0 - w_0\| = \sup_{w \in W} \|s_0 - w\| = g(s_0 - w_0) = \sup_{s \in S} \sup_{w \in W} \|s - w\|.$$

This implies that $s_0 \in F_S(W)$. $\square$

Now, we prove a new generalization of Theorem 1.13 in [8] in certain Banach spaces in the concept of simultaneous farthest points.

**Theorem 2.4.** *Let $X$ be a Banach space that has the mirror reflection property and $W$ be a non-empty convex remotal subset of $X$. If $S$ is a bounded subset of $X$ with $S \cap W = \phi$ and $s_0 \in S$, then the following are equivalent.*

(1) $s_0 \in F_S(W)$.

(2) *There exist $f \in X^*$, $w_0 \in F_W(s_0)$, and a subset $E \subseteq W$ with $\psi_{s_0}(E) \cap W \subset bd(B(s_0; r)$, where $r = \sup_{w \in W} \|s_0 - w\| = \|s_0 - w_0\|$ such that $f \in ext(B_{X^*})$ and*

$$f \in (\psi_{s_0}(E) - w_0)^0, \ f \in (S - s_0)^0, \tag{2.3}$$

$$f(s_0 - w_0) = \sup_{s \in S} \sup_{w \in W} \|s - w\|. \tag{2.4}$$

*Proof.*
(1) $\Rightarrow$ (2).
Let $s_0 \in F_S(W)$. By Theorem 2.3, there exists an $f \in X^*$ with $\|f\| = 1$ and $w_0 \in F_W(s_0)$ such that (2.1) and (2.2) are satisfied. Consider

$$M = \left\{ f \in X^* : \|f\| = 1, \ f(s_0 - w_0) = \sup_{s \in S} \sup_{w \in W} \|s - w\| \right\}.$$

Then by Theorem 2.3, $M \neq \phi$. Let $g \in M$ be such that $g = \lambda f_1 + (1 - \lambda) f_2$, $0 < \lambda < 1$ and $f_1, f_2 \in B_{X^*}$. Then

$$\sup_{s \in S} \sup_{w \in W} \|s - w\| = g(s_0 - w_0) = \lambda f_1(s_0 - w_0) + (1 - \lambda) f_2(s_0 - w_0).$$

Hence, since $0 < \lambda < 1$, and

$$|f_1(s_0 - w_0)| \leq \|s_0 - w_0\| = \sup_{s \in S} \sup_{w \in W} \|s - w\|,$$

$$|f_2(s_0 - w_0)| \leq \|s_0 - w_0\| = \sup_{s \in S} \sup_{w \in W} \|s - w\|,$$

we obtain

$$f_1(s_0 - w_0) = \sup_{s \in S} \sup_{w \in W} \|s - w\| = f_2((s_0 - w_0) = \|s_0 - w_0\|.$$

Consequently, $\|f_1\| = \|f_2\| = 1$ and $f_1, f_2 \in M$. Hence, $M$ is an extremal subset of $B_{X^*}$. Now let

$$N = M \cap (\psi_{s_0}(E) - w_0)^0 \cap (S - s_0)^0.$$

Then, by Theorem 2.3, $N \neq \phi$. Since $M$ is an extremal subset of $B_{X^*}$, it follows that $N$ is an extremal subset of $B_{X^*} \cap (\psi_{s_0}(E) - w_0)^0 \cap (S - s_0)^0$.

But $B_{X^*} \cap (\psi_{s_0}(E) - w_0)^0 \cap (S - s_0)^0$ is $w^*$-compact. Therefore, $N$ is $w^*$-compact. Hence, by the Krein-Milman Theorem, we conclude that $N \subset \text{clco}(\text{ext}(N))$. This implies

$$\phi \neq \text{ext}(N) = \text{ext}(B_{X^*}) \cap (\psi_{s_0}(E) - w_0)^0 \cap (S - s)^0 \cap N \subset \text{ext}(B_{X^*}) \cap N.$$

Let $f \in \text{ext}(N)$. Then $f \in \text{ext}(B_{X^*})$ and (2.3) and (2.4) are satisfied.

The implication (2) $\Rightarrow$ (1) is obvious. □

**Theorem 2.5.** *Let $X$ be a Banach space that has the mirror reflection property and $W$ be a non-empty convex remotal subset of $X$. If $S$ is a bounded subset of $X$ with $S \cap W = \phi$ and $s_0 \in S$, then the following are equivalent.*

(1) $s_0 \in F_S(W)$.

(2) *There exist $f \in X^*$, $w_0 \in F_W(s_0)$, and a subset $E \subseteq W$ with $\psi_{s_0}(E) \cap W \subseteq bd(B(s_0; r))$, where $r = \sup_{w \in W} \|s_0 - w\| = \|s_0 - w_0\|$ satisfying $f \in ext(B_{X_*})$ and*

$$|f(s_0 - w_0)| = \sup_{s \in S} \sup_{w \in W} \|s - w\|, \tag{2.5}$$

$$|f(s_0 - w_0)| \leq |f(s_0 - \psi_{s_0}(e))| \tag{2.6}$$

*for every $e \in E$.*

(3) *There exist $f \in X^*$, $w_0 \in F_W(s_0)$ and a subset $E \subseteq W$ with $\psi_{s_0}(E) \cap W \subseteq bd(B(s_0; r))$, where $r = \sup_{w \in W} \|s_0 - w\| = \|s_0 - w_0\|$ satisfying $f \in ext(B_{X^*})$, and*

$$f(s_0 - w_0) = \sup_{s \in S} \sup_{w \in W} \|s - w\| \tag{2.7}$$

$$f(s - s_0) f(s_0 - w_0) \leq 0, \quad \text{for every} \quad s \in S. \tag{2.8}$$

$$f(\psi_{s_0}(e) - s_0) f(s_0 - w_0) \leq 0, \quad \text{for every} \quad e \in E. \tag{2.9}$$

*Proof.*
(1) $\Rightarrow$ (2).
Assume that (1) holds. Then, by Theorem 2.4, there exist $f \in X^*$, $w_0 \in F_W(s_0)$, and a subset $E \subseteq W$ with $\psi_{s_0}(E) \cap W \subseteq \text{bd}(B(s_0; r))$, where $r = \sup_{w \in W} \|s_0 - w\| = \|s_0 - w_0\|$ such that $f \in \text{ext}(B_{X^*})$, and (2.3) and (2.4) are satisfied. Consequently, we obtain $|f(s_0 - w_0)| = \sup_{s \in S} \sup_{w \in W} \|s - w\|$. On the other hand, by (2.3) and (2.4) and since $f(\psi_{s_0}(e) - w_0) \leq 0$, we have:

$$\begin{aligned} |f(s_0 - w_0)| &= f(s_0 - w_0) \\ &= f(s_0 - \psi_{s_0}(e)) + f(\psi_{s_0}(e) - w_0) \\ &\leq f(s_0 - \psi_{s_0}(e)) \leq |f(s_0 - \psi_{s_0}(e))| \end{aligned}$$

for every $e \in E$.
(2) $\Rightarrow$ (1).
If we have (2), then we get

$$\begin{aligned}&\sup_{s\in S}\sup_{w\in W}\|s-w\| = |f(s_0-w_0)| \leq |f(s_0-\psi_{s_0}(e))| \\ &\leq \|f\|\,\|s_0-\psi_{s_0}(e)\| = \|s_0-\psi_{s_0}(e)\|,\end{aligned}$$

for every $e \in E$. Using Proposition 2.1, we get

$$\begin{aligned}&\sup_{s\in S}\sup_{w\in W}\|s-w\| \leq \inf_{e\in E}\|s_0-\psi_{s_0}(e)\|, = \sup_{e\in E}\|s_0-e\|, \\ &= \sup_{w\in W}\|s_0-w\| \leq \sup_{s\in S}\sup_{w\in W}\|s-w\|.\end{aligned}$$

Consequently,

$$\sup_{w\in W}\|s_0-w\| = \sup_{s\in S}\sup_{w\in W}\|s-w\|,$$

and $s_0 \in F_S(W)$.
(1) $\Rightarrow$ (3).
Assume now that (1) holds. Then, by Theorem 2.4, there exist $f \in X^*$, $w_0 \in F_W(s_0)$, and a subset $E \subseteq W$ with $\psi_{s_0}(E) \cap W \subset \mathrm{bd}(B(s_0;r))$, where $r = \sup_{w\in W}\|s_0-w\| = \|s_0 - w_0\|$ such that $f \in \mathrm{ext}(B_{X^*})$, and (2.3) and (2.4) are satisfied. Then by (2.4), we have

$$f(s_0-w_0) = \sup_{s\in S}\sup_{w\in W}\|s-w\| \geqslant 0.$$

Using (2.3), we have $f(s-s_0) \leq 0$ for every $s \in S$, and $f(\psi_{s_0}(e)-w_0) \leq 0$ for every $e \in E$. Consequently,

$$f(s-s_0)\,f(s_0-w_0) \leq 0,$$

for every $s \in S$ and

$$f(s-s_0)\,f(\psi_{s_0}(e)-w_0) \leq 0$$

for every $e \in E$.
(3) $\Rightarrow$ (1).
If (3) holds, then there exists $f \in X^*$ such that $f \in \mathrm{ext}(B_{X^*})$ satisfying (2.7), (2.8), and (2.9). Let $\omega = \mathrm{sign}(f(s_0-w_0))\,f$. Then by hypothesis, we have $\omega \in \mathrm{ext}(B_{X^*})$. Using (2.8) and (2.9), we have

$$\omega(s-s_0) = \mathrm{sign}(f(s_0-w_0))\,f(s-s_0) = \frac{f(s_0-w_0)}{|f(s_0-w_0)|}f(s-s_0) \leq 0$$

for every $s \in S$, and

$$\begin{aligned}\omega(\psi_{s_0}(e)-w_0) &= \mathrm{sign}(f(s_0-w_0))\,f(\psi_{s_0}(e)-w_0) \\ &= \frac{f(s_0-w_0)}{|f(s_0-w_0)|}f(\psi_{s_0}(e)-w_0) \leq 0\end{aligned}$$

for every $e \in E$. Furthermore,

$$\begin{aligned}\omega\left(s_0-w_0\right) &= \operatorname{sign}\left(f\left(s_0-w_0\right)\right) f\left(s_0-w_0\right) \\ &= \left|f\left(s_0-w_0\right)\right| = \sup_{s \in S} \sup_{w \in W} \|s-w\| .\end{aligned}$$

Hence, the functional $\omega$ satisfies (2) of Theorem 2.4 and $s_0 \in F_S(W)$. $\square$

**Theorem 2.6.** *Let $W$ be a convex remotal subset of a Banach space $X$ which has the mirror reflection property, and $S$ be a bounded set in $X$ with $S \cap W=\phi$. If $s_0 \in F_S(W)$ and $w_0 \in F_W\left(s_0\right)$, then there exists a subset $E \subseteq W$ such that $\psi_{s_0}(E)$ is convex and*

$$\left\|s_0-w_0\right\| = \max_{f \in ext(M)} f\left(s_0-w_0\right) = \inf_{e \in E} \max_{f \in ext(M)}\left|f\left(s_0-\psi_{s_0}(e)\right)\right| ,$$

*where $M=\left\{f \in X^*:\|f\|=1\right.$ and $\left.f\left(s_0-w_0\right)=\sup_{s \in S} \sup_{w \in W}\|s-w\|\right\}$.*

*Proof.* For every $y \in M$, let us denote by $\widetilde{y}$, the continuous bounded function on $\operatorname{ext}(M)$ defined by $\widetilde{y}(f)=f(y)$.

Let $\widetilde{X}$ be the space of all functions $\widetilde{y}\,(y \in X)$, endowed with the usual vector operations and with the norm

$$\|\widetilde{y}\|_{\widetilde{X}} = \max_{f \in \operatorname{ext}(M)}|\widetilde{y}(f)| ,$$

(that is, $\widetilde{X}$ is the image of $X$ in the space of all continuous and bounded functions on $\operatorname{ext}(M)$ under the mapping $y \longrightarrow \widetilde{y}$). By Theorem 2.4, the maximum is attained for $f_0 \in \operatorname{ext}(M)$. Two functions $\widetilde{y_1}, \widetilde{y_2} \in \widetilde{X}$ with $\widetilde{y_1}\left.\right|_{\operatorname{ext}(M)}=\widetilde{y_2}\left.\right|_{\operatorname{ext}(M)}$ are considered identical. Let $\widetilde{W}=\{\widetilde{w}: w \in W\}$, and $\widetilde{S}=\{\widetilde{s}: s \in S\}$. It is clear that $\widetilde{W}$ is a convex subset of $\widetilde{X}$ and $\widetilde{S}$ is a bounded subset of $\widetilde{X}$ such that $\widetilde{W} \cap \widetilde{S}=\phi$. Since $s_0 \in F_S(W)$, let $E$ be as in Theorem 2.4 and $g \in \operatorname{ext}\left(B_{X^*}\right)$ be a fixed linear functional with properties (2.3) and (2.4). Then,

$$g\left(s_0-\psi_{s_0}(e)\right)=g\left(s_0-w_0\right)+g\left(w_0-\psi_{s_0}(e)\right) \geqslant g\left(s_0-w_0\right)=\rho(W, S) ,$$

for every $e \in E$. Consequently, using (2.4), we have

$$\left\|\widetilde{s_0}-\widetilde{w_0}\right\|_{\widetilde{X}} \leq g\left(s_0-w_0\right)=\left|g\left(s_0-w_0\right)\right|=\left|\left(\widetilde{s_0}-\widetilde{w_0}\right)(g)\right| \leq\left\|\widetilde{s_0}-\widetilde{w_0}\right\|_{\widetilde{X}} .$$

Thus, we have

$$\left\|\widetilde{s_0}-\widetilde{w_0}\right\|_{\widetilde{X}}=g\left(s_0-w_0\right)=\left\|s_0-w_0\right\|=\rho(W, S) .$$

Therefore, using (2.4), we obtain

$$\begin{aligned}\inf_{e \in E}\left\|\widetilde{s_0}-\widetilde{\psi_{s_0}(e)}\right\|_{\widetilde{X}} &= \inf_{e \in E} \max_{f \in \operatorname{ext}(M)}\left|f\left(s_0-\psi_{s_0}(e)\right)\right| \\ &\geqslant \inf_{e \in E} g\left(s_0-\psi_{s_0}(e)\right)=g\left(s_0-w_0\right)=\left\|\widetilde{s_0}-\widetilde{w_0}\right\|_{\widetilde{X}} .\end{aligned}$$

This means that

$$\inf_{e\in E}\left\|\widetilde{s_0}-\widetilde{\psi_{s_0}(e)}\right\|_{\widetilde{X}} = \|\widetilde{s_0}-\widetilde{w_0}\|_{\widetilde{X}}.$$

Consequently,

$$\begin{aligned}\|s_0 - w_0\| &= \|\widetilde{s_0}-\widetilde{w_0}\|_{\widetilde{X}} = \max_{f\in \mathrm{ext}(M)}|f(s_0-w_0)| \\ &= \inf_{e\in E}\left\|\widetilde{s_0}-\widetilde{\psi_{s_0}(e)}\right\|_{\widetilde{X}} = \inf_{e\in E}\max_{f\in \mathrm{ext}(M)}|f(s_0-\psi_{s_0}(e))|.\end{aligned}$$

□

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