STRONG FORMS OF μ-LINDELÖFNESS WITH RESPECT TO HEREDITARY CLASSES

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ABSTRACT. The aim of this paper is to introduce and study strong forms of μ-Lindelöfness in generalized topological spaces with a hereditary class, called $\mathcal{S}\mu\mathcal{H}$-Lindelöfness and $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöfness. Interesting characterizations of these spaces are presented. Several effects of various types of functions on them are studied.

1. Introduction and Preliminaries

The idea of generalized topology and hereditary classes was introduced and studied by Császár in [1, 3], respectively. In this paper, we introduce and study strong forms of μ-Lindelöfness with respect to a hereditary class which was introduced by Qahis et al. in [8]. The strategy of using generalized topologies and hereditary classes to extend classical topological concepts have been used by many authors such as [3, 6, 10, 14].

Let X be a nonempty set and $p(X)$ the power set of X. A subfamily μ of $p(X)$ is called a generalized topology [1] if $\phi \in \mu$ and the arbitrary union of members of μ is again in μ. The pair (X, μ) is called a generalized topological space (briefly GTS). The elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A [1, 2]. A nonempty subcollection $\mathcal{H}$ of $p(X)$ is called a hereditary class (briefly HC) [3] if $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$. An HC $\mathcal{H}$ is called an ideal if $\mathcal{H}$ satisfies the additional condition: $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$ [7]. Some useful hereditary classes in X are: $p(A)$, where $A \subseteq X$, $\mathcal{H}_f$, the HC of all finite subsets of X, and $\mathcal{H}_c$, the HC of all countable subsets of X. We introduced the notion of $\mu\mathcal{H}$-Lindelöf spaces as follows [8]: A subset A of X is said to be $\mu\mathcal{H}$-Lindelöf if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of A by μ-open sets, there exists a countable subset Λ_0 of Λ such that $A \setminus \cup\{U_\lambda : \lambda \in \Lambda_0\} \in \mathcal{H}$. If $A = X$, then (X, μ) is called a $\mu\mathcal{H}$-Lindelöf space. A subset A of X is said to be μ-Lindelöf [12] if every cover of A by μ-open sets has a countable subcover. If $A = X$, then (X, μ) is called a μ-Lindelöf space. Given a generalized topological space (X, μ) with an HC $\mathcal{H}$, for a subset A of X, the generalized local function of A with respect to $\mathcal{H}$ and μ [3] is defined as follows: $A^*(\mathcal{H}, \mu) = \{x \in X : U \cap A \notin \mathcal{H} \text{ for all } U \in \mu_x\}$, where $\mu_x = \{U : x \in U \text{ and } U \in \mu\}$. If there is no confusion, we simply write A^* instead of $A^*(\mathcal{H}, \mu)$. And for a subset A of X, $c^*_\mu(A)$ is defined by $c^*_\mu(A) = A \cup A^*$. The family $\mu^* = \{A \subset X : X \setminus A = c^*_\mu(X \setminus A)\}$ is a GT on X which is finer than μ [3]. The elements of μ^* are called μ^*-open and the complement of a μ^*-open set is called a μ^*-closed set. It is clear that a subset A is μ^*-closed if and only if $A^* \subset A$. We call $(X, \mu, \mathcal{H})$ a hereditary generalized topological space and briefly we denote it by HGTS. If $(X, \mu, \mathcal{H})$ is an HGTS, the set $\mathcal{B} = \{V \setminus H : V \in \mu \text{ and } H \in \mathcal{H}\}$ is a base for a GT μ^*.

Definition 1.1. [11] *Let (X, μ) be a GTS. Then a subset A of X is called a μ-generalized closed set (in short, μg-closed set) if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and U is μ-open in X. The complement of a μg-closed set is called a μg-open set.*

Theorem 1.2. [3] *If (X, μ) is a GTS and $\mathcal{H}$ is a hereditary class on X, then for a subset A of X, $A^* \subset c_\mu(A)$.*

Theorem 1.3. [3] *Let (X, μ) be a GTS, $\mathcal{H}$ a hereditary class on X and A be a subset of X. If A is μ^*-open, then for each $x \in A$ there exist $U \in \mu_x$ and $H \in \mathcal{H}$ such that $x \in U \setminus H \subset A$.*

Definition 1.4. [1] *Let (X, μ) and (Y, ν) be two GTS's, then a function $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-continuous if $U \in \nu$ implies $f^{-1}(U) \in \mu$.*

Definition 1.5. *Let (X, μ) and (Y, ν) be two GTS's. A function $f : (X, \mu) \to (Y, \nu)$ is said to be:*

(1) *(μ, ν)-open (or μ-open)* [13] *if $U \in \mu$ implies $f(U) \in \nu$;*
(2) *(μ, ν)-closed (or μ-closed)* [11] *if $f(F)$ is ν-closed in Y for each μ-closed set F of X.*

2. $\mathcal{S}\mu\mathcal{H}$-Lindelöfness and $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöfness

In this section we define strong forms of $\mu\mathcal{H}$-Lindelöfness, called $\mathcal{S}\mu\mathcal{H}$-Lindelöf and $\mathbf{S}$-$\mathcal{S}\mu\mathcal{H}$-Lindelöf as follows.

Definition 2.1. *A subset A of GTS (X, μ) is said to be:*

(1) *$\mathcal{S}\mu\mathcal{H}$-Lindelöf if for every family $\{V_\alpha : \alpha \in \Lambda\}$ of μ-open sets such that $A \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{H}$, there exists a countable subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{H}$. If $A = X$, then (X, μ) is called an $\mathcal{S}\mu\mathcal{H}$-Lindelöf space;*

(2) ***S***-*$\mathcal{S}\mu\mathcal{H}$-Lindelöf if for every family $\{V_\alpha : \alpha \in \Lambda\}$ of μ-open sets such that $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$, there exists a countable subset Λ_0 of Λ such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. If $A = X$, then (X, μ) is called an* ***S***-*$\mathcal{S}\mu\mathcal{H}$-Lindelöf space.*

Now we can recall the classical definition of the Lindelöf property as follows. A topological space (X, τ) is called Lindelöf if every open cover of X has a countable subcover [7].

Remark 2.2. *The following properties are obvious from Definition 2.1:*

(1) *(X, μ) is μ-Lindelöf if and only if $(X, \mu, \{\phi\})$ is $\mathcal{S}\mu\{\phi\}$-Lindelöf.*
(2) *(X, μ) is μ-Lindelöf if and only if $(X, \mu, \{\phi\})$ is $\mathbf{S}-\mathcal{S}\mu\{\phi\}$-Lindelöf.*
(3) *The following diagram holds:*

$$\begin{array}{ccc} \boldsymbol{S}-\mathcal{S}\mu\mathcal{H}-\text{Lindelofness} & \Rightarrow & \mathcal{S}\mu\mathcal{H}-\text{Lindelofness} \\ \Downarrow & & \Downarrow \\ \mu-\text{Lindelofness} & \Rightarrow & \mu\mathcal{H}-\text{Lindelofness} \end{array}$$

(4) *Let $\mu = \tau$ be a topology. Then Lindelöfness on (X, τ) coincides with $\mathcal{S}\tau\{\emptyset\}$-Lindelöfness and $\mathbf{S}-\mathcal{S}\tau\{\emptyset\}$-Lindelofness on $(X, \tau, \{\emptyset\})$, where $\{\emptyset\}$ is a hereditary class.*

Next we will show that the implications above cannot be reversed.

Example 2.3. *Let $X = [0, +\infty)$, $\mu = \{X, (a, +\infty) : a \geq 0\} \cup \{\phi\}$, and $\mathcal{H} = \mathcal{H}_f$, then:*

(1) *$(X, \mu, \mathcal{H})$ is μ-Lindelöf. To prove this, let $\{V_\alpha : \alpha \in \Lambda\}$ be any an μ-cover of X, there exists $\alpha_0 \in \Lambda$ with $V_{\alpha_0} = X$, and so $X \setminus V_{\alpha_0} = \phi \in \mathcal{H}_f$.*
(2) *$(X, \mu, \mathcal{H})$ is not $\mathcal{S}\mu\mathcal{H}$-Lindelöf, because $X \setminus \cup\{(a, +\infty) : a > 0\} = \{0\} \in \mathcal{H}_f$, but if we let the increasing sequence $\{a_i : a_1 > 0, i \in \mathbb{Z}^+ - \{0\}\}$ then $X \setminus \cup\{(a_i, +\infty) : i \in \mathbb{Z}^+ - \{0\}\} = X \setminus (a_1, +\infty) \notin \mathcal{H}_f$.*

Example 2.4. *Let $X = \mathbb{R}^2$, μ is the Sorgenfrey topology $\mu = \{U \subseteq R^2 :$ for all $(x, y) \in U$, there exists $b > x$ and $c > y$ such that $[x, b) \times [y, c) \subseteq U\}$, and $\mathcal{H} = P(A)$. Now it is clear that (X, μ) is not μ-Lindelöf but it is evidently $\mathcal{S}\mu\mathcal{H}$-Lindelöf.*

Remark 2.5. *$\mathcal{S}\mu\mathcal{H}$-Lindelöfness and μ-Lindelöfness are independent of each other as Examples 2.3 and 2.4 show.*

Theorem 2.6. *An HGTS $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf if and only if for any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $\cap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$, there exists a countable subset Λ_0 of Λ such that $\cap\{F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$.*

Proof. Suppose that $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf. Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of μ-closed subsets of X such that $\cap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$. Then $\{X \setminus F_\alpha : \alpha \in \Lambda\}$ is a family of μ-open subsets of X. Let $H = \cap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$. Indeed,

$$X \setminus H = X \setminus \cap\{F_\alpha : \alpha \in \Lambda\} = \cup\{X \setminus F_\alpha : \alpha \in \Lambda\}.$$

Since $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $X \setminus \cup\{X \setminus F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$. This implies that $\cap\{F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$.

Conversely, let $\{V_\alpha : \alpha \in \Lambda\}$ be any family of μ-open subsets of X such that $X \setminus \cup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Then $\{X \setminus V_\alpha : \alpha \in \Lambda\}$ is a family of μ-closed subsets of X. By assumption we have $\cap\{X \setminus V_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$ and there exists a countable subset Λ_0 of Λ such that $\cap\{X \setminus V_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$. This implies that $X \setminus \cup\{V_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$. This shows that $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf. □

Theorem 2.7. *An HGTS $(X, \mu, \mathcal{H})$ is **S**-$\mathcal{S}\mu\mathcal{H}$-Lindelöf if and only if for any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $\cap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$, there exists a countable subset Λ_0 of Λ such that $\cap\{F_\alpha : \alpha \in \Lambda_0\} = \phi$.*

Proof. The proof is similar to that of Theorem 2.6 and is thus omitted. □

It is clear from the diagram that if X is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöf, then it is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, but the converse is not true as Example 2.4 shows.

A σ-ideal on a GTS (X, μ) is an ideal $\mathcal{H}$ which satisfies the following. If $\{A_i : i = 1, 2, 3, \ldots\} \subseteq \mathcal{H}$, then $\cup\{A_i : i = 1, 2, 3, \ldots\} \in \mathcal{H}$ (countable additivity).

Proposition 2.8. *If $(X, \mu, \mathcal{H})$ is an HGTS and $\mathcal{H}$ is a σ-ideal, then the following are equivalent:*

(1) *$(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf;*
(2) *$(X, \mu^*, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.*

Proof. (1) $\Rightarrow$ (2): Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ^*-open subsets of X such that $X \setminus \cup\{V_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$. For any $x \in \cup_{\alpha\in\Lambda} V_\alpha$, there exists $\alpha(x) \in \Lambda$ with $x \in V_{\alpha(x)}$. Then by Theorem 1.3, there exists $U_{\alpha(x)} \in \mu_x$ and $H_{\alpha(x)} \in \mathcal{H}$ such that $x \in U_{\alpha(x)} \setminus H_{\alpha(x)} \subset V_{\alpha(x)}$. And hence, $x \in U_{\alpha(x)} \subset \cup U_{\alpha(x)}$. Therefore, $\cup V_\alpha \subset \cup U_{\alpha(x)}$ and $X \setminus \cup U_{\alpha(x)} \subset X \setminus \cup V_\alpha \in \mathcal{H}$. Now $\{U_{\alpha(x)} : \alpha(x) \in \Lambda\}$ is a family of μ-open subsets of X and hence there exists a countable subset Λ_0 of Λ such that $X \setminus \bigcup_{\alpha(x)\in\Lambda_0} U_{\alpha(x)} \in \mathcal{H}$. It follows that

$$\begin{aligned} X \setminus \bigcup_{\alpha(x)\in\Lambda_0} U_{\alpha(x)} &\supset X \setminus \bigcup_{\alpha(x)\in\Lambda_0} \{(U_{\alpha(x)} \setminus H_{\alpha(x)}) \cup H_{\alpha(x)}\} \\ &= X \setminus [\bigcup_{\alpha(x)\in\Lambda_0} (U_{\alpha(x)} \setminus H_{\alpha(x)}) \cup (\bigcup_{\alpha(x)\in\Lambda_0} H_{\alpha(x)})] \\ &\supset X \setminus [\bigcup_{\alpha(x)\in\Lambda_0} V_{\alpha(x)} \cup (\bigcup_{\alpha(x)\in\Lambda_0} H_{\alpha(x)})]. \end{aligned}$$

Now set $H = X \setminus [\bigcup_{\alpha(x)\in\Lambda_0} V_{\alpha(x)} \cup (\bigcup_{\alpha(x)\in\Lambda_0} H_{\alpha(x)})]$. Since $\mathcal{H}$ is a σ-ideal, $\bigcup_{\alpha(x)\in\Lambda_0} H_{\alpha(x)} \in \mathcal{H}$ and also $H \cup (\bigcup_{\alpha(x)\in\Lambda} H_{\alpha(x)}) \in \mathcal{H}$. Observe that $X \setminus \cup\{V_{\alpha_x} : \alpha(x) \in \Lambda_0\} \subseteq H \cup (\bigcup_{\alpha(x)\in\Lambda} H_{\alpha(x)})$. In consequence

$$X \setminus \cup\{V_{\alpha(x)} : \alpha(x) \in \Lambda_0\} \in \mathcal{H}.$$

Thus, $(X, \mu^*, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.
(2) $\Rightarrow$ (1): Since $\mu \subseteq \mu^*$ we have that (2) $\Rightarrow$ (1). □

It is clear that if $(X, \mu, \mathcal{H})$ is an HGTS and $(X, \mu^*, \mathcal{H})$ is **S**-$\mathcal{S}\mu\mathcal{H}$-Lindelöf, then $(X, \mu, \mathcal{H})$ is **S**-$\mathcal{S}\mu\mathcal{H}$-Lindelöf.

Next we study the behavior of some types of subsets of $\mathcal{S}\mu\mathcal{H}$-Lindelöf and $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöf spaces.

Definition 2.9. [9] *A subset A of a HGTS $(X, \mu, \mathcal{H})$ is said to be $\mu\mathcal{H}_g$-closed if for every $U \in \mu$ with $A \setminus U \in \mathcal{H}$, $c_\mu(A) \subseteq U$.*

Theorem 2.10. *For a HGTS $(X, \mu, \mathcal{H})$, the following hold.*

1. *If $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf and $A \subseteq X$ is $\mu\mathcal{H}_g$-closed, then A is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.*
2. *If $(X, \mu, \mathcal{H})$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöf and $A \subseteq X$ is $\mu\mathcal{H}_g$-closed, then A is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöf.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since A is is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$. Then $(X \setminus c_\mu(A)) \cup \bigcup_{\alpha\in\Lambda} V_\alpha$ is a μ-covering of X and so $X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup_{\alpha\in\Lambda} V_\alpha)\right] = \phi \in \mathcal{H}$. Given that X is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that
$X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup_{\alpha\in\Lambda_0} V_\alpha)\right] \in \mathcal{H}$. But $X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup_{\alpha\in\Lambda_0} V_\alpha)\right] = c_\mu(A) \cap (X \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha) \supset A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Therefore, $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. Thus, A

is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.

(2) Let A be any $\mu\mathcal{H}_g$-closed of $(X,\mu,\mathcal{H})$ and $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since A is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \subseteq \bigcup\limits_{\alpha\in\Lambda} V_\alpha$. Then $(X \setminus c_\mu(A)) \cup (\bigcup\limits_{\alpha\in\Lambda} V_\alpha)$ is a μ-covering of X and hence, $X \setminus \left[(X \setminus c_\mu(A)) \cup (\bigcup\limits_{\alpha\in\Lambda} V_\alpha)\right] = \phi \in \mathcal{H}$. Given that X is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $X = (X \setminus c_\mu(A)) \cup (\bigcup\limits_{\alpha\in\Lambda_0} V_\alpha)$. Then $A = A \cap [(X \setminus c_\mu(A)) \cup \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha] = A \cap (\bigcup\limits_{\alpha\in\Lambda_0} V_\alpha) \subseteq \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$. Thus, A is $\mathbf{S}$-$\mathcal{S}\mu\mathcal{H}$-Lindelöf. □

Theorem 2.11. *For an HGTS $(X,\mu,\mathcal{H})$, the following hold.*

1. *If A and B are $\mathcal{S}\mu\mathcal{H}$-Lindelöf subsets of $(X,\mu,\mathcal{H})$ and $\mathcal{H}$ is an ideal, then $A \cup B$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.*
2. *If A and B are $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf subsets of $(X,\mu,\mathcal{H})$, then $A \cup B$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $(A \cup B) \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \subseteq A \cup B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \subseteq A \cup B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha$, then $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since A and B are $\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exist countable subsets Λ_0 and Λ_1 of Λ with $A \setminus \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda_1} V_\alpha \in \mathcal{H}$. This implies that $A \setminus \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha \in \mathcal{H}$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha \in \mathcal{H}$ and since $\mathcal{H}$ is an ideal we have that $(A \cup B) \setminus \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha = (A \setminus \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha) \cup (B \setminus \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha) \in \mathcal{H}$. Hence, $A \cup B$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.

(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $(A \cup B) \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \subseteq (A \cup B) \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \subseteq (A \cup B) \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha$, then $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and hence there exist countable subsets Λ_0 and Λ_1 of Λ such that $A \subseteq \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$ and $B \subseteq \bigcup\limits_{\alpha\in\Lambda_1} V_\alpha$. This implies that $A \subseteq \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha$ and $B \subseteq \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha$ and hence, $A \cup B \subseteq \bigcup\limits_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha$. Hence, $A \cup B$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf. □

The following example shows that the first part of the previous theorem does not hold when $\mathcal{H}$ is just a hereditary class, not an ideal.

Example 2.12. *Let $\mathbb{R}$ be the set of real numbers, μ the usual topology, $\mathcal{H} = \{A \subset \mathbb{R} : A \subset (0,1) \quad or \quad A \subset (1,2)\}$ and if $A = (0,1)$ and $B = (1,2)$, then:*
(1) It is clear that $A = (0,1)$ and $B = (1,2)$ are $\mathcal{S}\mu\mathcal{H}$-Lindelöf subsets.
(2) $A \cup B$ is not $\mathcal{S}\mu\mathcal{H}$-Lindelöf if $\{(a,+\infty) : \ a \geq 1\}$ is a family of μ-open subsets of X, $(A \cup B) \setminus \bigcup\limits_{a>1} (a,+\infty) = (A \cup B) \setminus (1,+\infty) \in \mathcal{H}$, but if k is a positive integer and $1 < a_1 < a_2 \cdots < a_k < \cdots$, then $(A \cup B) \setminus \bigcup\limits_{i=1}^{\infty} (a_i,+\infty) = (A \cup B) \setminus (a_1,+\infty) = (0,1) \cup (1,a_1) \notin \mathcal{H}$.

Theorem 2.13. *Let $(X,\mu,\mathcal{H})$ be an HGTS and A be a subset such that $A \setminus U \in \mathcal{H}$ for every $U \in \mu$. Then the following hold.*

1. *If there exists $B \subseteq X$ such that B is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, $A \subseteq B$ and $B \setminus U \in \mathcal{H}$, then A is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.*
2. *If there exists $B \subseteq X$ such that B is $\mathbf{S}$-$\mathcal{S}\mu\mathcal{H}$-Lindelöf, $A \subseteq B$ and $B \setminus U \in \mathcal{H}$, then A is $\mathbf{S}$-$\mathcal{S}\mu\mathcal{H}$-Lindelöf.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. There exists $B \subseteq X$ such that B is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, $A \subseteq B$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. There exists a countable subset Λ_0 of Λ such that $B \setminus \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. Since $A \setminus \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \subseteq B \setminus \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$, we have that $A \setminus \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$.

(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. There exists $B \subseteq X$ such that B is $\mathbf{S}$-$\mathcal{S}\mu\mathcal{H}$-Lindelöf, $A \subseteq B$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. There exists a countable subset Λ_0 of Λ with $B \subseteq \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$ and so $A \subseteq \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$.

□

Theorem 2.14. *Let $(X,\mu,\mathcal{H})$ be an HGTS and $A \subseteq B \subseteq c_\mu(A)$. Then the following hold.*

1. *Let A be $\mu\mathcal{H}g$-closed, then A is $\mathcal{S}\mu\mathcal{H}$-Lindelöf if and only if B is $\mathcal{S}\mu\mathcal{H}$-Lindelöf.*
2. *If A is μg-closed and $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf, then B is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf.*
3. *If A is $\mu\mathcal{H}g$-closed and B is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf, then A is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf.*

Proof. (1) Suppose that A is $\mathcal{S}\mu\mathcal{H}$-Lindelöf and $\{V_\alpha : \alpha \in \Lambda\}$ is a family of μ-open subsets of X such that $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. By the heredity property,

$A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and A is $\mathcal{S}\mu\mathcal{H}$-Lindelöf and hence there exists a countable subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. Since A is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and so $c_\mu(A) \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. This implies that $B \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$.
Conversely, suppose that B is $\mathcal{S}\mu\mathcal{H}$-Lindelöf and $\{V_\alpha : \alpha \in \Lambda\}$ is a family of μ-open subsets of X such $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Given that A is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = \phi \in \mathcal{H}$ and this implies $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since B is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $B \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. Hence, $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$.
(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and A is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Since A is μg-closed, $c_\mu(A) \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and this implies $B \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$.
(3) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Given that A is $\mu\mathcal{H}g$-closed, $c_\mu(A) \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = \phi \in \mathcal{H}$ and this implies $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since B is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset $\Lambda_0 \subseteq \Lambda$ with $B \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Hence, $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

3. Preservation by Functions

In this section, we study the behavior of $\mathcal{S}\mu\mathcal{H}$-Lindelöfness and $\mathbf{S}$-$\mathcal{S}\mu\mathcal{H}$-Lindelöfness under certain types of functions. First note that if $f : (X, \mu) \to (Y, \nu)$ and $\mathcal{H}$ is an HC on X, then $\mathcal{G} = \{B \subseteq Y : f^{-1}(B) \in \mathcal{H}\}$ is an HC on Y [9].

Theorem 3.1. *For a (μ, ν)-continuous function $f : (X, \mu) \to (Y, \nu)$, the following properties hold.*

1. *If $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, then $(Y, \nu, \mathcal{G})$ is $\mathcal{S}\nu\mathcal{G}$-Lindelöf.*
2. *If $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Lindelöf and f is a surjective function, then $(Y, \nu, \mathcal{G})$ is $\mathbf{S}-\mathcal{S}\nu\mathcal{G}$-Lindelöf.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of ν-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{G}$. Since $X \setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha\right) \in \mathcal{H}$ and $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ with

$f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha\right) = X \setminus \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \in \mathcal{H}$. Thus, $Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{G}$.

(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of ν-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{G}$. Since $X \setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha\right) \in \mathcal{H}$ and $(X, \mu, \mathcal{H})$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $X = \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)$. Given that f is surjective we have $Y = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

Lemma 3.2. [4] *Let $f : (X, \mu) \to (Y, \nu)$ be a function. If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ is a hereditary class on Y.*

Theorem 3.3. *For a bijective (μ, ν)-continuous function $f : (X, \mu) \to (Y, \nu)$, the following properties hold:*

1. *If $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, then $(Y, \nu, f(\mathcal{H}))$ is $\mathcal{S}\nu f(\mathcal{H})$-Lindelöf;*
2. *If $(X, \mu, \mathcal{H})$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöf, then $(Y, \nu, f(\mathcal{H}))$ is $\mathbf{S} - \mathcal{S}\nu f(\mathcal{H})$-Lindelöf.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of ν-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f(\mathcal{H})$. There exists $H \in \mathcal{H}$ with $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f(H)$. Then $H = f^{-1}(f(H)) = X \setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \in \mathcal{H}$. Given that $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha\right) = X \setminus \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \in \mathcal{H}$. Thus, $Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha = f(f^{-1}(Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha)) \in f(\mathcal{H})$.

(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of ν-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f(\mathcal{H})$. There exists $H \in \mathcal{H}$ such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f(H)$. Then $H = f^{-1}(f(H)) = X \setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \in \mathcal{H}$. Given that $(X, \mu, \mathcal{H})$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Lindelöf, there exists a countable subset Λ_0 of Λ such that $X = \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)$. Since f is surjective, $Y = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

It is clear that if $f : (X, \mu) \to (Y, \nu)$ and $\mathcal{G}$ is an HC on Y, then $f^{-1}(\mathcal{G}) = \{f^{-1}(G) : G \in \mathcal{G}\}$ is an HC on X.

Corollary 3.4. *Let $f : (X, \mu) \to (Y, \nu)$ be a bijective μ-open function. The following properties hold.*

1. *If $(Y, \nu, \mathcal{G})$ is $\mathcal{S}\nu\mathcal{G}$-Lindelöf, then $(X, \mu, f^{-1}(\mathcal{G}))$ is $\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf.*
2. *If $(Y, \nu, \mathcal{G})$ is $\mathbf{S}-\mathcal{S}\nu\mathcal{G}$-Lindelöf, then $(X, \mu, f^{-1}(\mathcal{G}))$ is $\mathbf{S}-\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f^{-1}(\mathcal{G})$. There exists $G \in \mathcal{G}$ with $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(G)$. Then $Y \setminus \bigcup_{\alpha\in\Lambda} f(V_\alpha) = f(f^{-1}(G)) = G \in \mathcal{G}$ and given that $(Y,\nu,\mathcal{G})$ is $\mathcal{S}\nu\mathcal{G}$-Lindelöf, then there exists a countable subset Λ_0 of Λ with $f(X \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha) = Y \setminus \bigcup_{\alpha\in\Lambda_0} f(V_\alpha) \in \mathcal{G}$. This implies that $X \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in f^{-1}(\mathcal{G})$.

(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f^{-1}(\mathcal{G})$. There exists $G \in \mathcal{G}$ with $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(G)$. Then $Y \setminus \bigcup_{\alpha\in\Lambda} f(V_\alpha) = f(f^{-1}(G)) = G \in \mathcal{G}$ and since $(Y,\nu,\mathcal{G})$ is $\mathbf{S}-\mathcal{S}\nu\mathcal{G}$- Lindelöf, then there exists a countable subset Λ_0 of Λ such that $Y = \bigcup_{\alpha\in\Lambda_0} f(V_\alpha)$. This implies that $X = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

Theorem 3.5. *Let $f : (X,\mu) \to (Y,\nu,\mathcal{G})$ be a μ-closed surjection.*

(1) *If for each $y \in Y$, $f^{-1}(y)$ is $\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X, then $f^{-1}(A)$ is $\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X whenever A is $\mathcal{S}\nu\mathcal{G}$-Lindelöf in Y.*
(2) *If for each $y \in Y$, $f^{-1}(y)$ is **S**-$\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X, then $f^{-1}(A)$ is **S**-$\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X whenever A is **S**- $\mathcal{S}\nu\mathcal{G}$-Lindelöf in Y.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $f^{-1}(A) \setminus \cup\{V_\alpha : \alpha \in \Lambda\} \in f^{-1}(\mathcal{G})$. For each $y \in A$ there exists a countable subset Λ_y of Λ such that $f^{-1}(y) \setminus \cup\{V_\alpha : \alpha \in \Lambda_y\} \in f^{-1}(\mathcal{G})$. Let $V_y = \cup\{V_\alpha : \alpha \in \Lambda_y\}$. Each V_y is a μ-open set in (X,μ) and $f^{-1}(y) \setminus V_y \in f^{-1}(\mathcal{G})$. Now each set $f(X - V_y)$ is ν-closed in Y and hence, $U(y) = Y - f(X - V_y)$ is a ν-open in (Y,ν) . Note that $f^{-1}(U(y)) \subseteq V_y$. Thus, $\{U(y) : y \in A\}$ is a family of ν-open subsets of Y such that $A \setminus \cup_{y\in A} U(y) \in \mathcal{H}$. Since A is $\mathcal{S}\nu\mathcal{G}$-Lindelöf in Y, there exists a countable subcollection $\{U(y_i) : \ i \in \mathbb{N}\}$ such that $A \setminus \cup\{U(y_i) : \ i \in \mathbb{N}\} \in \mathcal{G}$ and hence, $f^{-1}(A \setminus \cup\{U(y_i) : i \in \mathbb{N}\}) = f^{-1}(A) \setminus \cup\left\{f^{-1}(U(y_i)) : i \in \mathbb{N}\right\} \in f^{-1}(\mathcal{G})$. Since $f^{-1}(A) \setminus \cup\{V_{y_i} : i \in \mathbb{N}\} \subseteq f^{-1}(A) \setminus \cup\left\{f^{-1}(U(y_i)) : i \in \mathbb{N}\right\}$, then $f^{-1}(A) \setminus \cup\{V_{y_i} : i \in \mathbb{N}\} = f^{-1}(A) \setminus \cup\{V_\alpha : \alpha \in \Lambda_{y_i}, i \in \mathbb{N}\} \in f^{-1}(\mathcal{G})$. Hence, $f^{-1}(A)$ is $\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X.

(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $f^{-1}(A) \setminus \cup\{V_\alpha : \alpha \in \Lambda\} \in f^{-1}(\mathcal{G})$. Then it follows by assumption that for each $y \in A$ there exists a countable subset Λ_y of Λ such that $f^{-1}(y) \subseteq \cup\{V_\alpha : \alpha \in \Lambda_y\}$. Let $V_y = \cup\{V_\alpha : \alpha \in \Lambda_y\}$. Each V_y is a μ-open set in (X,μ) and $f^{-1}(y) \subseteq V_y$. Now each set $f(X - V_y)$ is ν-closed in Y and

$y \notin f(X - V_y)$ and hence, $U(y) = Y - f(X - V_y)$ is a ν-open set containing y. Note that $f^{-1}(U(y)) \subseteq V_y$. The collection $\{U(y) : y \in A\}$ is a family of ν-open sets of Y which covers A. Since A is $\mathbf{S}$-$\mathcal{S}\nu\mathcal{G}$-Lindelöf in Y, there exists a countable subcollection $\{U(y_i) : \ i \in \mathbb{N}\}$ such that $A \subseteq \cup\{U(y_i) : \ i \in \mathbb{N}\}$ and hence, $f^{-1}(A) \subseteq \cup\left\{f^{-1}(U(y_i)) : \ i \in \mathbb{N}\right\} \subseteq \cup\{V_{y_i} : \ i \in \mathbb{N}\}$, then

$$f^{-1}(A) \subseteq \cup\{V_{y_i} : \ i \in \mathbb{N}\} = \cup\{V_\alpha : \alpha \in \Lambda_{y_i}, i \in \mathbb{N}\}.$$

Hence, $f^{-1}(A)$ is $\mathbf{S}$-$\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X. □

Corollary 3.6. *Let $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ be a μ-closed function.*

1. *If $f^{-1}(y)$ is $\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X for each $y \in Y$ and (Y, ν) is $\mathcal{S}\nu\mathcal{G}$-Lindelöf, then (X, μ) is $\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf.*
2. *If $f^{-1}(y)$ is $\boldsymbol{S}$-$\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf in X for each $y \in Y$ and (Y, ν) is $\boldsymbol{S}$-$\mathcal{S}\nu\mathcal{G}$-Lindelöf, then (X, μ) is $\boldsymbol{S}$-$\mathcal{S}\mu f^{-1}(\mathcal{G})$-Lindelöf.*

A subset A of X is said to be μ-compact if for every μ-covering $\{U_\alpha : \alpha \in \Lambda\}$ of A there exists a finite subcollection $\{U_\alpha : \alpha \in \Lambda_0\}$ that also covers A [5].

Theorem 3.7. *Let $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ be a μ-closed surjection. If for each $y \in Y$, $f^{-1}(y)$ is μ-compact in X, then $f^{-1}(A)$ is $\mu f^{-1}(\mathcal{G})$-Lindelöf in X whenever A is $\nu\mathcal{G}$-Lindelöf in Y.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f^{-1}(A)$ by μ-open sets of X. For each $y \in A$ there exists a finite subset Λ_y of Λ such that $f^{-1}(y) \subseteq \cup\{V_\alpha : \alpha \in \Lambda_y\}$. Let $V(y) = \cup\{V_\alpha : \alpha \in \Lambda_y\}$. Each $V(y)$ is a μ-open set in (X, μ) and $f^{-1}(y) \subseteq V(y)$. Now each set $f(X - V(y))$ is ν-closed in Y and $y \notin f(X - V(y))$ hence, $U(y) = Y - f(X - V(y))$ is a ν-open set containing y. Note that $f^{-1}(U(y)) \subseteq V(y)$. The collection $\{U(y) : y \in A\}$ is a cover of A by ν-open sets of Y. Hence, there exists a countable subcollection $\{U(y_i) : i \in \mathbb{N}\}$ such that $A \setminus \cup\{U(y_i) : \ i \in \mathbb{N}\} \in \mathcal{G}$. Then $f^{-1}\left(A \setminus \cup\{U(y_i) : i \in \mathbb{N}\}\right) = f^{-1}(A) \setminus \cup\left\{f^{-1}\left(U(y_i)\right) : i \in \mathbb{N}\right\} \in f^{-1}(\mathcal{G})$. Since $f^{-1}(A) \setminus \cup\{V(y_i) : i \in \mathbb{N}\} \subseteq f^{-1}(A) \setminus \cup\left\{f^{-1}(U(y_i)) : \ i \in \mathbb{N}\right\}$, then $f^{-1}(A) \setminus \cup\{V(y_i) : i \in \mathbb{N}\} \in f^{-1}(\mathcal{G})$. Thus, $f^{-1}(A)$ is $\mu f^{-1}(\mathcal{G})$-Lindelöf in X. □

Corollary 3.8. *Let $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ be a μ-closed function such that $f^{-1}(y)$ is μ-compact in X for each $y \in Y$. If (Y, ν) is $\nu\mathcal{G}$-Lindelöf, then (X, μ) is $\mu f^{-1}(\mathcal{G})$-Lindelöf.*

Remark 3.9. *The above theorem and corollary remain valid if we assume that for each $y \in Y$, $f^{-1}(y)$ is μ-Lindelöf in X.*

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