GEOMETRY OF POLYNOMIALS WITH THREE ROOTS

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Abstract. Given a complex-valued polynomial of the form $p(z) = (z-1)^k(z-r_1)^m(z-r_2)^n$ with $|r_1| = |r_2| = 1$; $k, m, n \in \mathbb{N}$ and $m \neq n$, where are the critical points? The Gauss-Lucas Theorem guarantees that the critical points of such a polynomial will lie within the unit disk. This paper further explores the location and structure of these critical points. Surprisingly, the unit disk contains two 'desert' regions in which critical points cannot occur, and each c inside the unit disk and outside of the desert regions is the critical point of exactly two such polynomials.

1. Introduction

Given a complex-valued polynomial $p(z)$, the Gauss-Lucas Theorem implies that its critical points lie in the convex hull of its roots. If the roots of $p(z)$ form the vertices of a triangle, then its critical points must lie in that triangle. Several recent papers [1, 2, 3, 4, 6] have studied critical points of polynomials with three roots. If $p(z)$ is a complex-valued polynomial with roots r_1, r_2, and r_3, then there is a unique circle containing the roots. By changing coordinates, we can send this circle to the unit circle and fix $r_3 = 1$. The critical points of

$$\{p : \mathbb{C} \to \mathbb{C} \,|\, p(z) = (z-1)(z-r_1)(z-r_2),\ |r_1| = |r_2| = 1\}$$

are characterized in [1]. For this family of polynomials, a single critical point almost always determines a polynomial uniquely, and the unit disk contains a *desert*, $\{z \in \mathbb{C} : |z - \frac{2}{3}| < \frac{1}{3}\}$, in which critical points cannot occur.

For $k, m, n \in \mathbb{N}$, a natural extension of [1] is to study polynomials of the form

$$P(k, m, n) = \left\{p : \mathbb{C} \to \mathbb{C} \,|\, p(z) = (z-1)^k(z-r_1)^m(z-r_2)^n,\ |r_1| = |r_2| = 1\right\}.$$

Critical points of polynomials in $P(1, k, k)$ and $P(k, m, m)$ are characterized in [2] and [4], respectively. In either case, similar to [1], a critical point

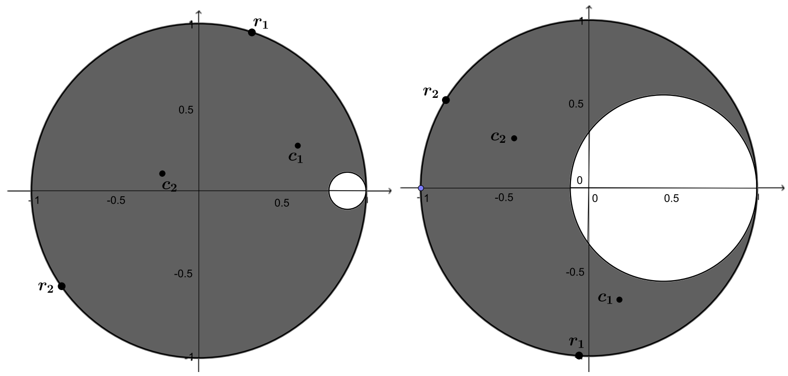

FIGURE 1. The desert region for $P(1, 4, 4)$ on the left, and $P(10, 4, 4)$ on the right.

almost always determines a polynomial uniquely, and the unit disk contains a desert in which critical points cannot occur. In Figure 1, the interior of the white disk is the desert region for $P(1, 4, 4)$ on the left, and $P(10, 4, 4)$ on the right. The structural similarity is due to the symmetry in the multiplicities of the roots located at r_1 and r_2.

This paper completes the characterization of the critical points of polynomials in $P(k, m, n)$ by analyzing the $m \neq n$ case. One can use *GeoGebra* to graphically investigate the critical points of these polynomials. Set r_1 and r_2 in motion around the unit circle and trace the loci of the critical points. See Figure 2. Due to the loss of symmetry in the multiplicities of the roots located at r_1 and r_2, the unit disk contains *two* desert regions in which critical points cannot occur. Furthermore, each c inside the unit disk and outside of the desert regions is the critical point of *exactly two* polynomials in $P(k, m, n)$.

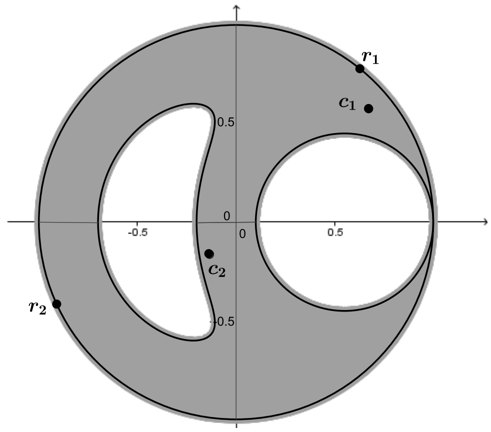

FIGURE 2. For the family of polynomials $P(9,3,8)$, the unit disk contains two desert regions in which critical points cannot occur.

2. Critical Points

In this paper, we study critical points of polynomials in $P(k,m,n)$ with $m \neq n$. Without loss of generality, we assume $m < n$. Our results can be applied to polynomials with three roots that lie elsewhere by changing the coordinate system.

We begin by introducing some notation. For $\alpha > 0$, let T_α denote the circle tangent to $x = 1$ with diameter α passing through 1 and $1-\alpha$ in the complex plane. That is,

$$T_\alpha = \left\{ z \in \mathbb{C} : \left| z - \left(1 - \frac{\alpha}{2}\right) \right| = \frac{\alpha}{2} \right\}.$$

For example, T_2 is the unit circle and $T_{\frac{2k}{m+n+k}}$ is a circle centered at $z = \frac{m+n}{m+n+k}$ with radius $\frac{k}{m+n+k}$. A given $z \in \mathbb{C}$ with $Re(z) \neq 1$ lies on a unique T_α. The following lemma provides a method of calculating the corresponding α-value.

Lemma 2.1 ([1]). *Let $z \in \mathbb{C}$ with $Re(z) \neq 1$. We have $z \in T_\alpha$ if and only if*

$$\frac{1}{\alpha} = Re\left(\frac{1}{1-z}\right).$$

A polynomial of the form

$$p(z) = (z-1)^k (z-r_1)^m (z-r_2)^n$$

with $|r_1| = |r_2| = 1$; $k, m, n \in \mathbb{N}$ and $n > m$ has $k + m + n - 1$ critical points: $k - 1$ critical points at $z = 1$, $m - 1$ critical points at $z = r_1$, $n - 1$ critical points at $z = r_2$, and two additional critical points in the unit disk. Differentiation gives

$$p'(z) = (z-1)^{k-1}(z-r_1)^{m-1}(z-r_2)^{n-1}q(z)$$

with

$$q(z) = (m+n+k)z^2 - ((k+n)r_1 + (k+m)r_2 + m + n)\, z + kr_1r_2 + nr_1 + mr_2.$$

Definition 2.2. *Given $p \in P(k,m,n)$, we say that c is a nontrivial critical point of $p(z)$ provided that $q(c) = 0$.*

Example 1. *Let $p \in P(k,m,n)$ have a nontrivial critical point at $z = 1$. Then, by the Gauss-Lucas Theorem, the root at $z = 1$ has multiplicity greater than k. Therefore, $p \in P(k,m,n)$ has a nontrivial critical point at $z = 1$ if and only if $p(z) = (z-1)^{k+m}(z-r)^n$ or $p(z) = (z-1)^{k+n}(z-r)^m$ for some $r \in T_2$.*

Since we know which $p \in P(k,m,n)$ have a nontrivial critical point at $z = 1$, we assume $c \neq 1$ as necessary throughout the paper.

To characterize the critical points of polynomials in $P(k,m,n)$, we investigate how the roots are related to a nontrivial critical point. Suppose c is a nontrivial critical point of $p(z) = (z-1)^k(z-r_1)^m(z-r_2)^n \in P(k,m,n)$. Then,

$$\begin{aligned} 0 &= q(c) \\ &= (k+m+n)c^2 - ((k+n)r_1 + (k+m)r_2 + m + n)\, c + kr_1r_2 + nr_1 + mr_2 \end{aligned}$$

and it follows that

$$r_1 = \frac{(m-(k+m)c)r_2 + (k+m+n)c^2 - (m+n)c}{-kr_2 + (k+n)c - n}$$

and

$$r_2 = \frac{(n-(k+n)c)r_1 + (k+m+n)c^2 - (m+n)c}{-kr_1 + (k+m)c - m}.$$

Definition 2.3. *Given $c \in \mathbb{C}$, we define*

$$f_{1,c}(z) = \frac{(m-(k+m)c)z + (k+m+n)c^2 - (m+n)c}{-kz + (k+n)c - n}$$

and

$$f_{2,c}(z) = \frac{(n-(k+n)c)z + (k+m+n)c^2 - (m+n)c}{-kz + (k+m)c - m}.$$

Furthermore, we let $S_1 = f_{1,c}(T_2)$ and $S_2 = f_{2,c}(T_2)$.

For $c \in \mathbb{C}$, $f_{1,c}$ and $f_{2,c}$ are Möbius transformations with $f_{1,c}(r_2) = r_1$ and $f_{2,c}(r_1) = r_2$. We have established the following theorem.

Theorem 2.4. *Suppose* $c \in \mathbb{C} \setminus \{1\}$ *and* $p(z) = (z-1)^k(z-r_1)^m(z-r_2)^n \in P(k,m,n)$. *Then, p has a nontrivial critical point at c if and only if* $f_{1,c}(r_2) = r_1$ *and* $f_{2,c}(r_1) = r_2$.

When $c = 1$, $f_{1,c}(z) = f_{2,c}(z) = \dfrac{-kz+k}{-kz+k} = 1$, and $f_{1,c}$ and $f_{2,c}$ are degenerate. When $c \neq 1$, $f_{1,c}$ and $f_{2,c}$ are invertible with

$$\begin{aligned}(f_{1,c})^{-1}(z) &= \frac{((k+n)c-n)z - ((k+m+n)c^2-(m+n)c)}{kz+m-(k+m)c} \\ &= \frac{(n-(k+n)c)z + (k+m+n)c^2-(m+n)c}{-kz+(k+m)c-m} \\ &= f_{2,c}(z).\end{aligned}$$

Furthermore, $f_{1,c}(r_2) = r_1 \in T_2$ so that $r_1 \in S_1 \cap T_2$, and $f_{2,c}(r_1) = r_2$ so that $r_2 \in S_2 \cap T_2$. To determine the polynomials in $P(k,m,n)$ having a critical point at c, we need to study $S_1 \cap T_2$ and $S_2 \cap T_2$. For $c \neq 1$, we will show that $|S_1 \cap T_2| = |S_2 \cap T_2|$ (Lemma 2.6) and that the cardinality of $S_1 \cap T_2$ determines the number of polynomials in $P(k,m,n)$ with a nontrivial critical point at c (Lemma 2.7).

As S_1 and T_2 are circles (or lines), $S_1 = T_2$ or $|S_1 \cap T_2| \leq 2$. To begin discussing the cardinality of the set $S_1 \cap T_2$ we need an additional fact related to Möbius transformations. Functions of the form $f(z) = e^{i\theta}\dfrac{z-\alpha}{\overline{\alpha}z-1}$ with $|\alpha| < 1$ are the only one-to-one analytic mappings of the unit disk onto itself [7, p. 334]. This leads to the following theorem.

Theorem 2.5. *A Möbius transformation T sends the unit circle to the unit circle if and only if* $T(z) = \dfrac{\bar{\alpha}z - \bar{\beta}}{\beta z - \alpha}$ *for some* $\alpha, \beta \in \mathbb{C}$ *with* $\left|\frac{\alpha}{\beta}\right| \neq 1$.

Example 2. *For* $c \in \mathbb{C}$, $S_1 = T_2$ *whenever* $f_{1,c}$ *satisfies Theorem 2.5. Since*

$$f_{1,c}(z) = \frac{(m-(k+m)c)z + (m+n+k)c^2-(m+n)c}{-kz+(k+n)c-n},$$

Theorem 2.5 implies $S_1 = T_2$ *if and only if*

$$\overline{(m+n)c-(m+n+k)c^2} = -k \quad \text{and} \quad \overline{m-(k+m)c} = n-(k+n)c.$$

The first equation reduces to $(m+n+k)c^2-(m+n)c-k = 0$ *so that*

$$0 = ((m+n+k)c+k)(c-1)$$

and $c \in \{1, \frac{-k}{m+n+k}\}$. *The hypothesis of Theorem 2.5 are not satisfied when* $c = 1$, *and as* $m \neq n$, $c = \frac{-k}{m+n+k}$ *does not satisfy* $\overline{m-(k+m)c} = n-(k+n)c$. *Therefore, there is no* $c \in \mathbb{C}$ *for which* $S_1 = T_2$.

Remark 1. *When $n = m$, $f_{1,c} = f_{2,c}$ which implies $S_1 = S_2$. In this case, $S_1 = T_2$ precisely when $c = \frac{-k}{m+n+k}$. See* [4].

The following two Lemmas are direct extensions of results in [3].

Lemma 2.6 ([3])**.** *If $c \neq 1$, then $|S_1 \cap T_2| = |S_2 \cap T_2| \in \{0, 1, 2\}$.*

Proof. Let $c \neq 1$. By Example 2, $S_1 \neq T_2$, and as $(f_{1,c})^{-1} = f_{2,c}$, it follows that $S_2 \neq T_2$. Without loss of generality, suppose $|S_2 \cap T_2| = 1$ and $S_1 \cap T_2 = \{\alpha, \beta\}$ with $\alpha \neq \beta$. By the definition of S_1, there exist $\alpha_0, \beta_0 \in T_2$ with $f_{1,c}(\alpha_0) = \alpha$, $f_{1,c}(\beta_0) = \beta$ and $\alpha_0 \neq \beta_0$. This implies

$$f_{2,c}(\alpha) = \alpha_0 \quad \text{and} \quad f_{2,c}(\beta) = \beta_0$$

so that $|S_2 \cap T_2| > 1$; a contradiction. Therefore, $|S_1 \cap T_2| = |S_2 \cap T_2|$. $\square$

Lemma 2.7 ([3])**.** *Suppose $n \neq m$ and $c \in \mathbb{C} \setminus \{1\}$.*

1. *If S_1 and T_2 are disjoint, then no $p \in P(k, m, n)$ has a critical point at c.*
2. *If S_1 and T_2 are tangent, then c is the nontrivial critical point of exactly one $p \in P(k, m, n)$.*
3. *If S_1 and T_2 intersect in two distinct points, then c is the nontrivial critical point of exactly two polynomials in $P(k, m, n)$.*

Lemmas 2.6 and 2.7 imply that S_1 is sufficient to characterize the nontrivial critical points of $p \in P(k, m, n)$.

2.1. **Center and Radius of S_1.** According to Lemma 2.7, to further characterize critical points of $p \in P(k, m, n)$, we need a better understanding of S_1. Since

$$f_{1,c}(z) = \frac{(m - (k+m)c)z + (k+m+n)c^2 - (m+n)c}{-kz + (k+n)c - n},$$

S_1 is a line whenever there exists a $z \in T_2$ with $-kz + (k+n)c - n = 0$. That is,

$$k = |kz| = |(k+n)c - n| \quad \longleftrightarrow \quad \frac{k}{n+k} = \left| c - \frac{n}{n+k} \right|. \tag{2.1}$$

Therefore, S_1 is a line if and only if $c \in T_{\frac{2k}{n+k}}$.

Example 3. *Let $c \in T_{\frac{2k}{n+k}}$. Then S_1 is a line passing through*

$$f_{1,c}(1) = \frac{(m+n+k)c - m}{n+k} \quad \textit{and} \quad f_{1,c}(-1) = \frac{(m+n+k)c^2 - (n-k)c - m}{(n+k)c - (n-k)}$$

with

$$f_{1,c}(1) - f_{1,c}(-1) = \frac{-2mnc + 2mn}{(n+k)((n+k)c - (n-k))}. \tag{2.2}$$

Since $c \in T_{\frac{2k}{n+k}}$, $c = \frac{n}{n+k} + \frac{k}{n+k}e^{i\theta}$ for some $\theta \in [0, 2\pi]$. Substituting into equation (2.2), we obtain $Re(f_{1,c}(1) - f_{1,c}(-1)) = 0$. Therefore, S_1 is a vertical line through $f_{1,c}(1)$. For future use, we observe that if $c \in T_{\frac{2k}{n+k}}$, then $f_{1,c}(1) \in T_{\frac{2k(m+n+k)}{(n+k)^2}}$.

For $c \in \mathbb{C} \setminus \{1\}$, we use methods from [3] and [4] to determine the center and radius of S_1. By the definition of S_1, $z \in S_1$ if and only if there exists a $w \in T_2$ with $f_{1,c}(w) = z$. As $(f_{1,c})^{-1} = f_{2,c}$, it follows that $z \in S_1$ if and only if $|f_{2,c}(z)| = |w| = 1$. That is,

$$\left|\frac{(n-(k+n)c)z + (m+n+k)c^2 - (m+n)c}{-kz + (k+m)c - m}\right| = 1$$

which implies

$$\begin{aligned}&\left|z - \left(\frac{(k+m)c-m}{k}\right)\right| \\ &= \left|\frac{n-(n+k)c}{k}\right| \left|z - \left(\frac{(m+n)c - (m+n+k)c^2}{n-(n+k)c}\right)\right|.\end{aligned}$$

Applying the change of variables $z = W + f_{1,c}(1) = W + \frac{(m+n+k)c-m}{n+k}$ gives

$$\left|W - \frac{mn(c-1)}{k(n+k)}\right| = \left|\frac{n-(n+k)c}{k}\right| \left|W - \frac{mn(1-c)}{(n+k)(n-(n+k)c)}\right|. \quad (2.3)$$

For $\lambda \neq 1$, it follows from introductory complex analysis that the solution set of

$$|z-u| = \lambda\,|z-v|$$

is a circle with center $C = v + \dfrac{v-u}{\lambda^2 - 1}$ and radius R satisfying $R^2 = |C|^2 - \dfrac{\lambda^2|v|^2 - |u|^2}{\lambda^2 - 1}$. In (2.3), when $\lambda = |\frac{n-(n+k)c}{k}| = 1$, equation (2.1) implies $c \in T_{\frac{2k}{n+k}}$ and by Example 3, S_1 is a line. When $c \notin T_{\frac{2k}{n+k}}$, it follows that $|\frac{n-(n+k)c}{k}| \neq 1$ and the solution set of (2.3) is a circle with center

$$\begin{aligned}C &= \frac{mn(1-c)}{(n+k)(n-(n+k)c)} + \frac{\frac{mn(1-c)}{(n+k)(n-(n+k)c)} + \frac{mn(1-c)}{k(n+k)}}{\left|\frac{n-(n+k)c}{k}\right|^2 - 1} \\ &= \frac{\left(\frac{1}{k}\right)^2 |n-(n+k)c|^2 \frac{mn(1-c)}{(n+k)(n-(n+k)c)} + \frac{mn(1-c)}{k(n+k)}}{\left|\frac{n-(n+k)c}{k}\right|^2 - 1}.\end{aligned}$$

Using $|n-(n+k)c|^2 = (n-(n+k)c)(\overline{n-(n+k)c})$ yields

$$C = \frac{\frac{mn}{n+k}((\overline{n-(n+k)c})(1-c)+k(1-c))}{|n-(n+k)c|^2-k^2}$$
$$= \frac{mn|1-c|^2}{|n-(n+k)c|^2-k^2}$$
$$= \frac{mn}{\left|n+k-\frac{k}{1-c}\right|^2-\left|\frac{k}{1-c}\right|^2}.$$

By Lemma 2.1, $\frac{k}{1-c} = \frac{k}{\alpha} + iky$ for some real value y, and it follows that

$$C = \frac{\alpha mn}{(n+k)^2\alpha - 2k(n+k)}.$$

Moreover,

$$R^2 = |C|^2 - \frac{\left|\frac{n-(n+k)c}{k}\right|^2\left|\frac{mn(1-c)}{(n+k)(n-(n+k)c)}\right|^2-\left|\frac{mn(c-1)}{k(n+k)}\right|^2}{\left|\frac{n-(n+k)c}{k}\right|^2-1} = |C|^2$$

implies $R = |C|$. Resubstituting $z = W + \frac{(m+n+k)c-m}{n+k}$ establishes the following result.

Lemma 2.8. *Let $c \in T_\alpha \setminus \{1\}$ with $\alpha \neq \dfrac{2k}{n+k}$. Then, S_1 is a circle with center γ and radius r given by*

$$\gamma = \frac{(m+n+k)c-m}{n+k} + \frac{\alpha mn}{(n+k)^2\alpha - 2k(n+k)}$$
$$\textit{and } r = \left|\frac{\alpha mn}{(n+k)^2\alpha - 2k(n+k)}\right|.$$

We investigate an example for future reference.

Example 4. *For $c \in T_2 \setminus \{1\}$, S_1 is a circle with center*

$$\gamma = \frac{(m+n+k)c-m}{n+k} + \frac{2mn}{2(n+k)^2-2k(n+k)} = \frac{m+n+k}{n+k}c$$

and radius

$$r = \left|\frac{2mn}{2(n+k)^2-2k(n+k)}\right| = \frac{m}{n+k}.$$

Therefore, when $c \in T_2 \setminus \{1\}$, S_1 is externally tangent to T_2 at c.

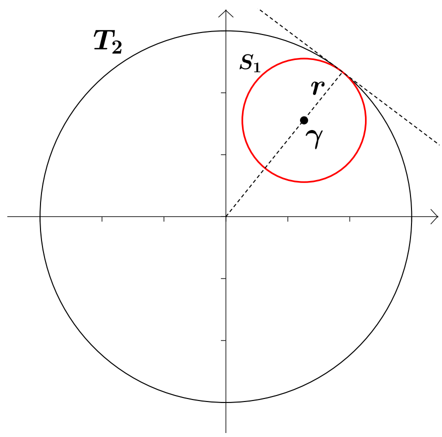

FIGURE 3. If S_1 is internally tangent to T_2, then $|\gamma| + r = 1$.

2.2. **Identifying the Desert Regions.** For $p \in P(k, m, n)$ with $n > m$, we use methods from [3] to locate the desert regions. According to Lemma 2.7, if c is located within a desert region, then $S_1 \cap T_2 = \emptyset$. To better understand the desert regions we identify their boundaries by determining where S_1 is tangent to T_2. We begin with S_1 internally tangent to T_2. For $c \in T_\alpha \setminus \{1\}$ with $\alpha \in (0, 2]$, if S_1 is internally tangent to T_2, then

$$|\gamma| + r = 1. \tag{2.4}$$

See Figure 3.

For $1 \neq c \in T_\alpha$ and $R = \dfrac{\alpha mn}{\alpha(n+k)^2 - 2k(n+k)}$, S_1 is a circle with center $\gamma = \dfrac{(m+n+k)c - m}{n+k} + R$ and radius $r = |R|$. Substituting into equation (2.4) and setting $c = x + iy$ yields

$$((m+n+k)x - m + (n+k)R)^2 + (m+n+k)^2\, y^2 = (n+k)^2(1 - |R|)^2. \tag{2.5}$$

We let I_α denote the set of all (x, y) which satisfy equation (2.5). Since $r > 0$, equation (2.4) is satisfied if and only if S_1 is internally tangent to T_2 or $S_1 = T_2$. Recalling, from Example 2, that S_1 is never equal to T_2, we obtain the following lemma.

Lemma 2.9. *Let $c \neq 1$ and $\alpha \in (0, 2]$. Then, S_1 is internally tangent to T_2 if and only if $c \in I_\alpha \cap T_\alpha$.*

To use Lemma 2.9, we need to find the values of α for which I_α and T_α intersect, and the corresponding points of intersection. Observe that R

is undefined when $\alpha = \frac{2k}{n+k}$, negative when $\alpha < \frac{2k}{n+k}$, and positive when $\alpha > \frac{2k}{n+k}$. With this in mind, we consider three cases:

(1) $0 < \alpha < \frac{2k}{n+k}$;
(2) $\alpha = \frac{2k}{n+k}$;
(3) $\frac{2k}{n+k} < \alpha \leq 2$.

In the first case, $|R| = -R$ and equation (2.5) becomes

$$\left(x - \left(1 - \frac{(n+k)+(n+k)R}{m+n+k}\right)\right)^2 + y^2 = \left(\frac{(n+k)+(n+k)R}{m+n+k}\right)^2. \tag{2.6}$$

For

$$\rho = \frac{(n+k)+(n+k)R}{m+n+k} = \frac{((n+k)^2+nm)\alpha - 2k(n+k)}{(n+k)(m+n+k)\alpha - 2k(m+n+k)}, \tag{2.7}$$

I_α is a circle tangent to $x = 1$, centered at $x = 1-\rho$ with radius $|\rho|$. Circles I_α and T_α intersect (at $1 \neq c$) precisely when $I_\alpha = T_\alpha$. This occurs when

$$\frac{((n+k)^2+mn)\alpha - 2k(n+k)}{(n+k)(m+n+k)\alpha - 2k(m+n+k)} = \frac{\alpha}{2}.$$

After simplification, this becomes

$$((m+n+k)\alpha - 2k)((n+k)\alpha - 2(n+k)) = 0,$$

which implies $\alpha = \dfrac{2k}{m+n+k}$ or $\alpha = 2 \notin \left(0, \frac{2k}{n+k}\right)$. By Lemma 2.9, when $c \in T_{\frac{2k}{m+n+k}}$, S_1 is internally tangent to T_2.

Before proceeding to the second case, we pause for a result.

Theorem 2.10. *No polynomial in $P(k,m,n)$ has a critical point strictly inside $T_{\frac{2k}{m+n+k}}$.*

Proof. Let $c \in T_\alpha \setminus \{1\}$ with $0 < \alpha < \frac{2k}{m+n+k}$. Equations (2.6) and (2.7) imply $I_\alpha = T_\beta$ with

$$\beta = \frac{2[((n+k)^2+mn)\alpha - 2k(n+k)]}{(n+k)(m+n+k)\alpha - 2k(m+n+k)}.$$

Furthermore, $\beta = \alpha$ when $\alpha = \frac{2k}{m+n+k}$ or $\alpha = 2$, $\beta > 0$ when $\alpha = 0$, and β is undefined when $\alpha = \frac{2k}{n+k}$. Therefore, if $0 < \alpha < \frac{2k}{m+n+k}$, then $\beta > \alpha$ and it follows that T_α lies inside $T_\beta = I_\alpha$. See Figure 4. For $c = x+iy$, equation (2.6) implies

$$\left(x - \left(1 - \frac{(n+k)+(n+k)R}{m+n+k}\right)\right)^2 + y^2 < \left(\frac{(n+k)+(n+k)R}{m+n+k}\right)^2.$$

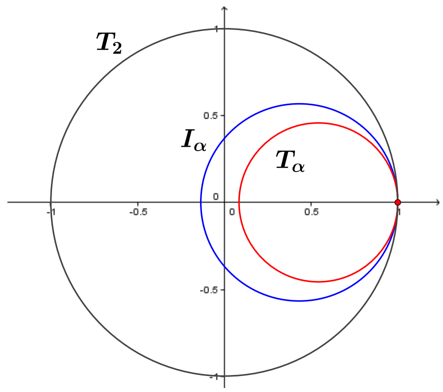

FIGURE 4. When $0 < \alpha < \frac{2k}{m+n+k}$, T_α lies inside I_α.

Equivalently, equations (2.4) and (2.5) imply $|\gamma| + r < 1$. Therefore, $S_1 \cap T_2 = \emptyset$ and by Lemma 2.7, no $p \in P(k,m,n)$ has a critical point strictly inside $T_{\frac{2k}{m+n+k}}$. $\square$

In the second case, $\alpha = \frac{2k}{n+k}$ and by Example 3, S_1 is a vertical line passing through $f_{1,c}(1) = \frac{(m+n+k)c-m}{n+k}$ which is not tangent to T_2.

In the third case, $|R| = R$ and equation (2.5) becomes

$$\left(x - \left(\frac{m-n-k}{m+n+k} + \frac{(n+k)-(n+k)R}{m+n+k}\right)\right)^2 + y^2 = \left(\frac{(n+k)-(n+k)R}{m+n+k}\right)^2.$$

For

$$\rho = \frac{(n+k)-(n+k)R}{m+n+k} = \frac{((n+k)^2 - mn)\alpha - 2k(n+k)}{(n+k)(m+n+k)\alpha - 2k(m+n+k)},$$

I_α is a circle tangent to $x = \frac{m-n-k}{m+n+k}$, centered at $x = \dfrac{m-n-k}{m+n+k} + \rho$ with radius $|\rho|$. Furthermore, I_α and T_α intersect on the real axis if and only if

$$\frac{m-n-k}{m+n+k} + \frac{2(n+k)-2(n+k)R}{m+n+k} = 1-\alpha \quad \text{or} \quad \frac{m-n-k}{m+n+k} = 1-\alpha. \tag{2.8}$$

The second equation implies $\alpha = \frac{2(n+k)}{m+n+k}$ and the first equation reduces to

$$(n+k)(m+n+k)\alpha^2 - (2k(m+n+k) + 2mn)\alpha = 0$$

which has solutions $\alpha = \frac{2(m+k)}{m+n+k}$ and $\alpha = 0$. We set

$$\alpha_1 = \frac{2(m+k)}{m+n+k} < \frac{2(n+k)}{m+n+k} = \alpha_2.$$

When $\alpha \in (\frac{2k}{n+k}, \alpha_1) \cap (\alpha_2, 2]$, $I_\alpha \cap T_\alpha = \emptyset$ and by Lemma 2.9, S_1 is not internally tangent to T_2.

When $\alpha_1 < \alpha < \alpha_2$, $|I_\alpha \cap T_\alpha| = 2$. See Figure 5. To determine the values of c where S_1 is internally tangent to T_2, we need to find the intersection of I_α and T_α. Upon simplification, these equations become:

$$\left(x - \frac{m-(n+k)R}{m+n+k}\right)^2 + y^2 = \left(\frac{(n+k)-(n+k)R}{m+n+k}\right)^2$$
$$\alpha(1-x) - (1-x)^2 = y^2.$$

By setting $R = \dfrac{\alpha mn}{(n+k)^2\alpha - 2k(n+k)}$ and using substitution, we eventually obtain

$$x = \frac{(m+n+k)^2\alpha^2 - \left[2(m+n+k)(m+n+2k) - 4mn\right]\alpha + 4k(m+n+k)}{(m+n+k)((m+n+k)\alpha - 2k)(\alpha-2)} \tag{2.9}$$

and

$$y^2 = (1-x)(\alpha - 1 + x).$$

As α varies from α_1 to α_2, a parametric curve is formed. See Figure 5. After some tedious algebra the parametric equations combine to form the implicit equation

$$2k(m+n)((-1+x)x+y^2)(-1+x^2+y^2) + k^2(-1+x^2+y^2)^2 + ((-1+x)^2 + y^2)(m^2(-1+x^2+y^2) + n^2(-1+x^2+y^2) + 2mn(1+x^2+y^2)) = 0. \tag{2.10}$$

Equation (2.10) represents the boundary of the second desert region which we denote by D.

Theorem 2.11. *No polynomial in $P(k,m,n)$ has a critical point strictly inside D.*

Proof. Let $c = x+iy \in T_\alpha$ with $\alpha \in [\alpha_1, \alpha_2]$. Then, c lies inside D whenever

$$\left(x - \frac{m-(n+k)R}{m+n+k}\right)^2 + y^2 < \left(\frac{(n+k)-(n+k)R}{m+n+k}\right)^2.$$

Equivalently, equations (2.4) and (2.5) imply $|\gamma| + r < 1$. Therefore, S_1 and T_2 are disjoint and by Lemma 2.7, c is not the critical point of any $p \in P(k,m,n)$. □

Remark 2. *When $n = m$, direct calculations give $\alpha_1 = \alpha_2$. This observation explains why the family $P(k,m,m)$ has only one desert region. See* [4].

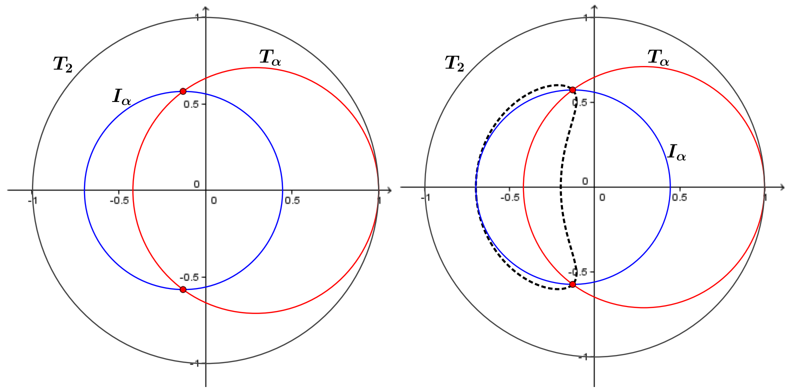

FIGURE 5. Left, when $\alpha_1 < \alpha < \alpha_2$, $|I_\alpha \cap T_\alpha| = 2$. Right, as α varies from α_1 to α_2, parametric equations (2.9) trace the boundary of the second desert region.

Example 5. *Setting $k = 1$, $m = 1$, and $n = 2$ in Theorem 2.10 and equation (2.10) identifies*

$$2(x^2 + y^2)^2 - 3x(x^2 + y^2) + x = 0$$

and $T_{\frac{1}{2}}$ as the boundaries of the desert regions in $P(1, 1, 2)$, which constitutes the results of [3]. *See Figure 6.*

The *GeoGebra* notebook used to create the images in Figure 6 is located at `http://www.uwplatt.edu/~frayerc/deserts.html`. One can use this to explore the boundary of the desert regions for different values of k, m, or n!

The analysis of I_α has established the following result.

Lemma 2.12. *Let $c \in \mathbb{C} \setminus \{1\}$. Then, S_1 is internally tangent to T_2 if and only if $c \in T_{\frac{2k}{m+n+k}} \cup D$.*

Furthermore, for $c \in T_\alpha$ with $0 < \alpha \leq 2$, S_1 will be externally tangent to T_2 if and only if

$$|\gamma| - r = 1.$$

A less involved analysis establishes the following lemma.

Lemma 2.13. *Let $c \in \mathbb{C} \setminus \{1\}$. Then, S_1 is externally tangent to T_2 if and only if $c \in T_2$.*

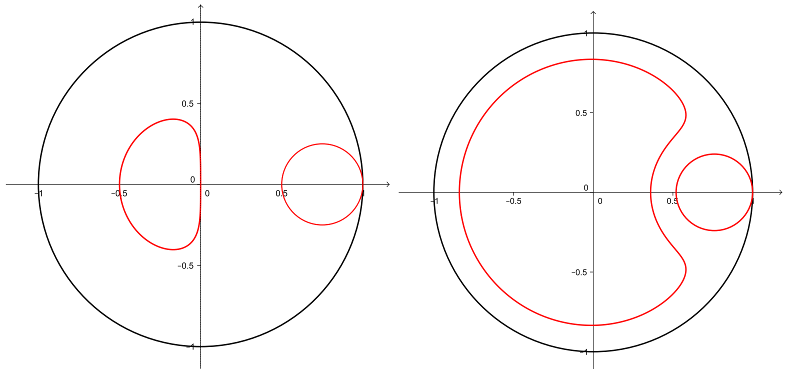

FIGURE 6. The boundary of the desert regions in $P(1,1,2)$, to the left, and $P(6,2,17)$ to the right.

2.3. **Main Result.** For $n \neq m$, we are now ready to characterize the critical points of polynomials in $P(k,m,n)$. Let O represent the region strictly inside T_2 and outside of $T_{\frac{2k}{m+n+k}}$ and D. Observe that O is the interior of the grey region in Figure 2. We denote the closure of O by $\overline{O}$.

Theorem 2.14. *Let $c \in \mathbb{C}$ and $n \neq m$.*

(1) *For $p \in P(k,m,n)$, p has a nontrivial critical point at $c = 1$ if and only if $p(z) = (z-1)^{k+m}(z-r)^n$ or $p(z) = (z-1)^{k+n}(z-r)^m$ for some $r \in T_2$.*
(2) *If $c \notin \overline{O}$, then no $p \in P(k,m,n)$ has a critical point at c.*
(3) *If $1 \neq c \in \overline{O} \setminus O$, then a unique $p \in P(k,m,n)$ has a nontrivial critical point at c.*
(4) *If $c \in O$, then exactly two polynomials in $P(k,m,n)$ have a nontrivial critical point at c.*

Proof. Parts 1–3 follow directly from Example 1 and Lemmas 2.7, 2.12, and 2.13.

For part 4, we use a 'root dragging' argument similar to [1]. By Lemma 2.12 and Example 2, for $c \in O$, $|S_1 \cap T_2| \in \{0,2\}$. We will show that $|S_1 \cap T_2| = 2$. Without loss of generality, suppose $S_1 \cap T_2 = \emptyset$ with S_1 contained inside T_2. As we 'drag' c to T_2 along a smooth curve contained in O, S_1 is continuously transformed into a circle externally tangent to T_2. By the Intermediate Value Theorem, there exists a c_0 on the curve with S_1 internally tangent to T_2. As c never leaves O, this contradicts Lemma 2.12. Therefore, $|S_1 \cap T_2| = 2$ and by Lemma 2.7, there are exactly two polynomials in $P(k,m,n)$ with a nontrivial critical point at c. □

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