INTEGER INVARIANTS OF AN INCIDENCE MATRIX RELATED TO ROTA'S BASIS CONJECTURE

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ABSTRACT. We compute the spectrum and Smith normal form of the incidence matrix of disjoint transversals, a combinatorial object closely related to the n-dimensional case of Rota's basis conjecture.

1. INTRODUCTION AND DEFINITIONS

Let n be a positive integer, and consider a square array of n^2 distinct elements:

$$
\begin{array}{cccc}
a_1 & a_2 & \cdots & a_n \\
b_1 & b_2 & \cdots & b_n \\
\vdots & \vdots & \vdots & \vdots \\
c_1 & c_2 & \cdots & c_n.
\end{array}
\tag{1.1}
$$

A transversal of the array (1.1) is an n-element set consisting of exactly one element from each row. Two transversals are said to be disjoint if and only if they are disjoint as sets. We write $\mathcal{T}_n$ to denote the set of all transversals, ordered in some fashion.

These transversals together with the relation of disjointness will be the object of our study. We can encode this information into an incidence matrix (more accurately, an adjacency matrix) as follows. Define the *incidence matrix of disjoint transversals* A_n to be the $n^n \times n^n$ matrix whose rows and columns are indexed by $\mathcal{T}_n$ such that the (i, j)-entry of A_n is one if the ith transversal and the jth transversal are disjoint and zero otherwise.

The spectrum of A_n does not depend on how the set of transversals was ordered, and so we may view it as an invariant of the incidence relation. More fundamental is the *Smith normal form* of A_n, which is unchanged even under independent row and column permutations of A_n and hence describes the incidence relation at a more basic level.

Recall that the Smith normal form of a (possibly nonsquare) integer matrix is a diagonal matrix of the same size, with the diagonal entries

subject to certain divisibility conditions. More formally, if A is an $m \times n$ integer matrix, then there exist unimodular (invertible over the integers) matrices P and Q such that the matrix $PAQ = (d_{i,j})$ satisfies

$$d_{i,j} = 0, \text{ for } i \neq j$$

and

$$d_{i,i} \text{ divides } d_{i+1,i+1}, \text{ for } 1 \leq i < \min\{m, n\}.$$

This diagonal matrix is called the Smith normal form of A. These diagonal entries $d_{i,i}$ are unique up to sign, and are called the invariant factors of the matrix A.

In the next section, we will compute the spectrum and the Smith normal form of A_n, the incidence matrix of disjoint transversals. The matrices A_n are actually the association matrices for the maximal distance in the Hamming association schemes $H(n, n)$, and their eigenvalues are known [4]. We give another computation of the spectrum because it is relevant to how we will calculate the Smith normal form of A_n. Our main result was conjectured in [2] and first established in [9] using character-theoretic methods. Here we give an elementary proof. In the final section, we explain the connection between the matrix A_n and Rota's basis conjecture.

2. The Spectrum and Smith Normal Form of A_n

Theorem 2.1. *Let A_n be the incidence matrix of disjoint transversals.*

(1) *The eigenvalues of A_n are*

$$(-1)^{n-k}(n-1)^k$$

occurring with multiplicity

$$\binom{n}{k}(n-1)^{n-k}$$

for $0 \leq k \leq n$.

(2) *The invariant factors of A_n are*

$$(n-1)^k$$

occurring with multiplicity

$$\binom{n}{k}(n-1)^{n-k}$$

for $0 \leq k \leq n$.

Proof. Let $V = \mathbb{R}^{\mathfrak{T}_n}$, the vector space over the reals consisting of all formal linear combinations of elements of $\mathfrak{T}_n$. Then A_n represents the linear map $\psi : V \to V$ defined by

$$\psi(\{a_{i_1}, b_{i_2}, \cdots, c_{i_n}\}) = \sum_{a_{j_1} \neq a_{i_1}, \cdots, c_{j_n} \neq c_{i_n}} \{a_{j_1}, b_{j_2}, \cdots, c_{j_n}\}, \qquad (2.1)$$

for $\{a_{i_1}, b_{i_2}, \cdots, c_{i_n}\} \in \mathfrak{T}_n$. That is, ψ is the linear map defined by sending a transversal to the sum of the transversals disjoint to it.

Next, let $[n]$ denote the set $\{1, 2, \cdots, n\}$ and define $W = \mathbb{R}^{[n]}$, the vector space over the reals consisting of all formal linear combinations of elements of $[n]$. If B_n is the $n \times n$ matrix whose diagonal entries are zero and all other entries are one, then B_n represents the linear map $\theta : W \to W$ defined by

$$\theta(i) = \sum_{j \in [n], j \neq i} j, \qquad (2.2)$$

for $i \in [n]$. That is, θ sends an element in $[n]$ to the (formal) sum of all the elements of $[n]$ distinct from it. It is easy to see that $n - 1$ is an eigenvalue of B_n with multiplicity one, and -1 is an eigenvalue with multiplicity $n - 1$ (eigenvectors will have entries that sum to zero).

Now, in order to see the connection between (2.1) and (2.2), let us take the tensor product of W with itself n times and consider the induced linear map $\theta^{\otimes n} : W^{\otimes n} \to W^{\otimes n}$ defined by

$$\theta^{\otimes n}(i_1 \otimes \cdots \otimes i_n) = \theta(i_1) \otimes \cdots \otimes \theta(i_n).$$

Since the tensor product respects the distributive law over vector addition, we have that

$$\begin{aligned} \theta^{\otimes n}(i_1 \otimes \cdots \otimes i_n) &= \theta(i_1) \otimes \cdots \otimes \theta(i_n), \\ &= \Big(\sum_{j_1 \neq i_1} j_1\Big) \otimes \cdots \otimes \Big(\sum_{j_n \neq i_n} j_n\Big), \\ &= \sum_{j_1 \neq i_1, \cdots, j_n \neq i_n} j_1 \otimes \cdots \otimes j_n. \end{aligned} \qquad (2.3)$$

It is easy to see that the vector spaces V and $W^{\otimes n}$ are isomorphic under the map that sends a transversal $\{a_{i_1}, b_{i_2}, \cdots, c_{i_n}\}$ to the simple tensor $i_1 \otimes i_2 \otimes \cdots \otimes i_n$, and after identifying these spaces the maps (2.1) and (2.3) are the same. In other words, A_n can be obtained by taking the Kronecker product of B_n with itself n times. Therefore, part (1) of the theorem is proved if we show that the eigenvalues of $\theta^{\otimes n}$ are

$$(-1)^{n-k}(n-1)^k$$

occurring with multiplicity

$$\binom{n}{k}(n-1)^{n-k}$$

for $0 \le k \le n$. To see this, take a basis of eigenvectors $\{x_i\}_{i=1}^n$ for the map $\theta : W \to W$ in (2.2). Let x_1 have eigenvalue $n-1$ and let $x_2, x_3, \cdots, x_n$ each have eigenvalue -1. Let $x_{i_1} \otimes \cdots \otimes x_{i_n}$ be an arbitrary basis vector for $W^{\otimes n}$. Then it is easy to see that this basis vector is in fact an eigenvector for $\theta^{\otimes n}$ and both the eigenvalue and multiplicity depend on how many times x_1 occurs in this simple tensor. Explicity, if x_1 occurs exactly k times in the simple tensor $x_{i_1} \otimes \cdots \otimes x_{i_n}$ then

$$\begin{aligned}
\theta^{\otimes n}(x_{i_1} \otimes \cdots \otimes x_{i_n}) &= \theta(x_{i_1}) \otimes \cdots \otimes \theta(x_{i_n}) \\
&= (-1)^{n-k}(n-1)^k x_{i_1} \otimes \cdots \otimes x_{i_n},
\end{aligned}$$

so $(-1)^{n-k}(n-1)^k$ is an eigenvalue of $\theta^{\otimes n}$. Additionally, the number of basis vectors containing a factor of x_1 exactly k times is $\binom{n}{k}(n-1)^{n-k}$. This completes the proof of part (1).

Next, let us prove part (2) of the theorem. It is amazing that a nearly identical argument goes through to give the invariant factors of A_n. Set $V = \mathbb{Z}^{\mathcal{T}_n}$ and $W = \mathbb{Z}^{[n]}$, the free modules over the integers consisting of all formal linear combinations of elements of $\mathcal{T}_n$ and $[n]$, respectively. The maps (2.1) and (2.2) should now be viewed as homomorphisms of free abelian groups. It is an easy exercise using integral row and column operations to show that B_n has Smith normal form:

$$S = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & n-1 \end{bmatrix}.$$

Therefore, 1 is an invariant factor with multiplicity $n-1$ and $n-1$ is an invariant factor with multiplicity 1. This means that we can find bases $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ of W so that

$$\theta(x_1) = (n-1)y_1$$
$$\theta(x_i) = y_i,$$

for $2 \le i \le n$. The matrix A_n still represents the homomorphism $\theta^{\otimes n} : W^{\otimes n} \to W^{\otimes n}$, and the sets $\{x_{i_1} \otimes \cdots \otimes x_{i_n}\}$ and $\{y_{i_1} \otimes \cdots \otimes y_{i_n}\}$ are bases of $W^{\otimes n}$. If x_1 occurs exactly k times in the basis vector $x_{i_1} \otimes \cdots \otimes x_{i_n}$,

then we have

$$\theta^{\otimes n}(x_{i_1} \otimes \cdots \otimes x_{i_n}) = \theta(x_{i_1}) \otimes \cdots \otimes \theta(x_{i_n}),$$
$$= (n-1)^k y_{i_1} \otimes \cdots \otimes y_{i_n}.$$

Hence, the invariant factors of A_n are

$$(n-1)^k$$

occurring with multiplicity

$$\binom{n}{k}(n-1)^{n-k}$$

for $0 \leq k \leq n$, and this completes the proof. $\qquad\square$

3. Relationship to Rota's Basis Conjecture

The following conjecture was made by Gian-Carlo Rota in 1989 [7].

Suppose one has n bases $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n$ of an n-dimensional vector space $\mathcal{V}$. Form an $n \times n$ array, with the elements of $\mathcal{B}_i$ forming the ith row. Then Rota's basis conjecture states that there is a way to independently permute the entries of each row so that all the columns of the array are also bases of $\mathcal{V}$.

This conjecture is very general, stated for any finite dimensional vector space over any field. The basis conjecture is implied for even dimensions and fields of specific characteristics (in particular, characteristic zero) by the Alon-Tarsi conjecture concerning even and odd Latin squares [1, 8]. Indeed, much of the recent progress on Rota's basis conjecture has been through investigations of Latin squares [5, 6]. The conjecture also generalizes to an identical statement about bases in a rank n matroid that has been fully settled only for $n \leq 3$ [3].

If we assign the elements of the ith basis $\mathcal{B}_i$ to the positions in the ith row of the array (1.1), then Rota's basis conjecture asserts that there will exist n mutually disjoint transversals where each transversal corresponds to a basis of $\mathcal{V}$. Now, which transversals will correspond to bases will depend on the original choice of $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n$, and each such choice will distinguish a principal submatrix of A_n. Thus in this sense the matrix A_n contains the information needed to verify any particular instance of Rota's basis conjecture. It is hoped that the matrix A_n may offer a new approach to this problem.

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