CENTERS AND GENERALIZED CENTERS OF NEARRINGS WITHOUT IDENTITY

G. ALAN CANNON, VINCENT GLORIOSO, BRAD BAILEY HALL, AND TAYLOR TRICHE

Abstract. The center of a nearring N, in general, is not a subnearring of N. The center, however, is contained in a related structure, the generalized center, which is always a subnearring. We give three constructions of nearrings without multiplicative identity and characterize their centers and generalized centers. We find that the centers of these nearrings are always subnearrings.

1. Introduction

A (right) nearring is a triple $(N, +, \cdot)$ where $(N, +)$ is a group, written additively though not necessarily abelian, $(N, \cdot)$ is a semigroup, and the right distributive law holds, i.e., $(a + b)c = ac + bc$ for all $a, b, c \in N$. So nearrings are generalizations of rings. The books [6] and [7] provide comprehensive information on nearrings.

The center of a nearring N is $C(N) = \{c \in N \mid cx = xc \text{ for all } x \in N\}$. The center of a ring R is always a subring of R. However, the center of a nearring N may not be a subnearring of N as the following example shows.

Example 1.1. *Let $N = (\mathbb{Z}_3[x], +, \circ)$, the nearring of polynomials in one variable over the ring $\mathbb{Z}_3$ under usual polynomial addition and composition. Then N has identity x, and $x \in C(N)$. Let $p(x) = 2x \in N$. It follows that $p \circ x^2 = 2x^2$ and $x^2 \circ p = (2x)^2 = x^2$. So $x + x = 2x = p \notin C(N)$, and $C(N)$ is not a subnearring of N.*

If we consider the set of distributive elements of a nearring N, $N_d = \{d \in N \mid d(a + b) = da + db \text{ for all } a, b \in N\}$, and define the generalized center of N, $GC(N) = \{x \in N \mid xd = dx \text{ for all } d \in N_d\}$, then the set $GC(N)$ is always a subnearring of N. Throughout the paper, we will use the facts that $C(N) \subseteq N_d$ and $C(N) \subseteq GC(N)$.

Centers and generalized centers of nearrings were first studied in [5] and [2]. Further work appeared in [3], where necessary and sufficient conditions for $C(N)$ to be a subnearring were found for various classes of nearrings with identity. In [4], the GAP package SONATA [1] was used to generate examples of nearrings without identity in which the centers were subnearrings. General constructions were found to describe these specific examples.

In this paper, we continue the work in [4]. We present three constructions of nearrings in which the centers are subnearrings. These nearrings, in general, do not have multiplicative identities. We also determine and compare their centers and generalized centers and provide many examples to illustrate the theory. These constructions might also be useful in other investigations of nearrings.

2. TS Nearrings

This first construction appeared in [4]. This version, however, includes hypotheses not considered previously. Part (1) of the theorem indicates the result in [4], while part (2) is new.

Theorem 2.1. *Let $(G,+)$ be a finite group of even order, not necessarily abelian. Suppose there exists $\emptyset \neq T \subseteq G^*$ such that $G\backslash T$ is a (normal) subgroup of G of index 2 and there is $\emptyset \neq S \subseteq T$ with $S = S_1 \dot{\cup} S_2 \dot{\cup} \cdots \dot{\cup} S_n$, a partition of S. Further suppose that for $1 \leq i \leq n$, there are distinct $0 \neq q_i \in G$ with $|q_i| = 2$ such that either (1) $q_i \in G\backslash T$ for all i or (2) $S = T$ and $q_i \in S_i$ for all i.*

Define a multiplication on G by

$$a \cdot b = \begin{cases} q_1, & \text{if } a \in T, b \in S_1; \\ q_2, & \text{if } a \in T, b \in S_2; \\ \vdots & \\ q_n, & \text{if } a \in T, b \in S_n; \\ 0, & \text{otherwise.} \end{cases}$$

Then $N = (G,+,\cdot)$ is a right, zero-symmetric nearring.

Proof. The proof of (1) appears in [4]. For (2), since $a0 = 0a = 0$ for all $a \in N$, N is zero-symmetric. We now need to show that multiplication is associative and distributes over addition from the right.

To show associativity, let $a,b,c \in N$. If $c \notin S$ or $a \notin T$, then $(ab)c = 0 = a(bc)$. So assume $c \in S_i$ for some i and $a \in T$. If $b \in S_j$, then $(ab)c = q_j c = q_i = aq_i = a(bc)$. If $b \notin S$, then $(ab)c = 0c = 0 = a0 = a(bc)$. Hence, multiplication is associative.

For distributivity, let $a,b,c \in N$. If $c \notin S = T$, then $(a+b)c = 0 = 0+0 = ac+bc$. Thus, assume $c \in S_i$. There are four cases to consider. If $a,b \notin T$, then $a+b \notin T$. So $(a+b)c = 0 = 0+0 = ac+bc$. If $a \in T$ and $b \notin T$, then $a+b \in T$. So $(a+b)c = q_i = q_i + 0 = ac+bc$. If $a \notin T$ and $b \in T$, then $a+b \in T$. So $(a+b)c = q_i = 0+q_i = ac+bc$. If $a,b \in T$, then

$a+b \notin T$. So $(a+b)c = 0 = q_i + q_i = ac + bc$. It follows that the right distributive law holds. □

We call the nearring above a *TS nearring*. We note that in [4], TS nearrings only referred to the nearrings constructed via (1). Since the construction is similiar for (2), we use the same name.

The following results appear in [4] for part (1) of Theorem 2.1. We include them here, without proof, for completeness.

Theorem 2.2. *Let N be a TS nearring with $q_i \in G\backslash T$ for all $1 \leq i \leq n$. Then N does not have a multiplicative identity. Furthermore,*

(1) *If $n = 1$ and $S = T$, then $C(N) = N_d = GC(N) = N$, making N a commutative nearring.*
(2) *If $n = 1$ and $S \subsetneq T$, then $N \setminus T = C(N) = N_d \subsetneq GC(N) = N$.*
(3) *If $n \geq 2$, then $N \setminus T = C(N) = N_d \subsetneq GC(N) = N$.*

In all cases, $C(N)$ is a subnearring of N.

Example 2.3. *Let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$, the quaternion group of order 8, and $T = \{\pm j, \pm k\}$. Then $G\backslash T = \{\pm 1, \pm i\}$ is a subgroup of G of index 2. Let $S = S_1 = \{\pm j\} \subsetneq T$ and $q_1 = -1$. Then by part (2) of Theorem 2.2, $N \setminus T = C(N) = N_d \subsetneq GC(N) = N$.*

We now turn our attention to the second type of TS nearring.

Theorem 2.4. *Let N be a TS nearring with $S = T$ and $q_i \in S_i$ for all $1 \leq i \leq n$. Then N has a multiplicative identity if and only if $N \cong \mathbb{Z}_2$. Furthermore,*

(1) *If $n = 1$, then $C(N) = N_d = GC(N) = N$.*
(2) *If $n \geq 2$, then $C(N) = N\backslash T = N_d$ and $GC(N) = N$.*

In all cases, $C(N)$ is a subnearring of N.

Proof. Suppose the nearring N has multiplicative identity, 1. Let $b \notin T$. Then $b = 1b = 0$ and $G\backslash T = \{0\}$. Since $G\backslash T$ is a subgroup of G of index 2, then $|T| = |G\backslash T| = 1$ and $T = \{1\} = S = S_1$. Constructing the addition and multiplication tables for N using these sets and $q_1 = 1$ yields $N \cong \mathbb{Z}_2$. The reverse implication is clear.

Assume $n = 1$. Then $S = T = S_1$. We show that $N \subseteq C(N)$. Let $x \in N$. If $x \notin T = S$, then for any $y \in N$, $xy = 0 = yx$ and $x \in C(N)$. If $x \in T = S = S_1$, we consider two cases for $y \in N$. If $y \in T = S = S_1$, then $xy = q_1 = yx$. If $y \notin T = S$, then $xy = 0 = yx$. So $x \in C(N)$. We conclude that $N \subseteq C(N)$.

Since $C(N) \subseteq N_d \subseteq N \subseteq C(N)$, it follows that $C(N) = N_d = N$. Furthermore, since $N_d = C(N)$, it is clear that $GC(N) = N$, and the result follows.

Now assume $n \geq 2$. Let $d \in N_d$. Assume $d \in T$. Choose $a \in S_1$ and $b \in S_2$. Since $a, b \in T$, then $a + b \notin T$. Hence, $0 = d(a + b) = da + db = q_1 + q_2$. So $q_1 = q_2$, a contradiction. Thus, $d \notin T$. It follows that $N_d \subseteq N \backslash T$.

Let $c \in N \backslash T$. For $x \in N$, $cx = 0 = xc$. So $c \in C(N)$ and $N \backslash T \subseteq C(N)$. Combining this inequality with the one above yields $C(N) \subseteq N_d \subseteq N \backslash T \subseteq C(N)$. Thus all sets are equal.

Since $N_d = C(N)$, it is clear that $GC(N) = N$. This completes the proof. $\square$

The following examples illustrate both cases of Theorem 2.4.

Example 2.5. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $S = S_1 = T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{1\}$. Then $G \backslash T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$ is a subgroup of G of index 2. Also let $q_1 = (1, 1, 1) \in S_1$. Then by part (1) of the previous theorem, $C(N) = N_d = GC(N) = N$.*

Example 2.6. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $S = T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{1\}$. Then $G \backslash T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$ is a subgroup of G of index 2. Let $q_1 = (1, 1, 1) \in S_1 = \{(1, 1, 1)\}$, $q_2 = (1, 0, 1) \in S_2 = \{(1, 0, 1)\}$, $q_3 = (0, 1, 1) \in S_3 = \{(0, 1, 1)\}$, and $q_4 = (0, 0, 1) \in S_4 = \{(0, 0, 1)\}$. Then by part (2) of the previous theorem, $C(N) = N \backslash T = N_d$ and $GC(N) = N$.*

3. A Modification of Complemented Malone Nearrings

Complemented Malone nearrings were defined in [4].

Let $(G, +)$ be an abelian group and suppose $\emptyset \neq S \subseteq G^*$ such that for all $x \in S$, $x \notin -S$. Define a multiplication on G by

$$a \cdot b = \begin{cases} a, & \text{if } b \in S; \\ -a, & \text{if } b \in -S; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper we modify the definition. While the multiplication still depends upon a set S_1 and its negative $-S_1$, we also include a second set S_2. The inclusion of this second set requires the abelian group G to have even order and a subgroup H of index two.

Theorem 3.1. *Let G be a finite, abelian group of even order with a subgroup H of index 2. Also let $\emptyset \neq S_1, S_2 \subseteq G^*$ where if $x \in S_1$, then $x \notin -S_1$, $S_2 = -S_2$, and $G = H \,\dot\cup\, S_1 \,\dot\cup\, (-S_1) \,\dot\cup\, S_2$. Let $q \in S_2$ with $|q| = 2$.*

Define a multiplication on G by

$$a \cdot b = \begin{cases} a, & \text{if } b \in S_1; \\ -a, & \text{if } b \in -S_1; \\ q, & \text{if } a \notin H, b \in S_2; \\ 0, & \text{otherwise.} \end{cases}$$

Then $N = (G, +, \cdot)$ is a right, zero-symmetric nearring without multiplicative identity.

Proof. Since $0a = a0 = 0$ for all $a \in N$, N is zero-symmetric. For associativity, let $a, b, c \in N$. If $c \in H$, then $(ab)c = 0 = a0 = a(bc)$. If $c \in S_1$, then $(ab)c = ab = a(bc)$. If $c \in -S_1$, then $(ab)c = -(ab)$ and $a(bc) = a(-b)$. We now consider cases for the element b. If $b \in S_1$, then $-(ab) = -a = a(-b)$. If $b \in -S_1$, then $-(ab) = -(-a) = a = a(-b)$. If $b \in H$, then $-(ab) = 0 = a(-b)$. If $b \in S_2$ and $a \in H$, then $-(ab) = 0 = a(-b)$. If $b \in S_2$ and $a \notin H$, then $-(ab) = -q = q = a(-b)$.

If $c \in S_2$, we consider several cases. If $b \in H$, then $(ab)c = 0c = 0 = a0 = a(bc)$. If $b \notin H$, we consider the cases of $a \in H$ or $a \notin H$. Assume $a \in H$. If $b \in S_1$, then $(ab)c = ac = 0 = aq = a(bc)$. If $b \in -S_1$, then $(ab)c = (-a)c = 0 = aq = a(bc)$. If $b \in S_2$, then $(ab)c = 0c = 0 = aq = a(bc)$. Now assume $a \notin H$. If $b \in S_1$, then $(ab)c = ac = q = aq = a(bc)$. If $b \in -S_1$, then $(ab)c = (-a)c = q = aq = a(bc)$. If $b \in S_2$, then $(ab)c = qc = q = aq = a(bc)$. We have exhausted all cases and multiplication in N is associative.

For right distributivity, let $a, b, c \in N$. If $c \in H$, then $(a + b)c = 0 = 0 + 0 = ac + bc$. If $c \in S_1$, then $(a + b)c = a + b = ac + bc$. If $c \in -S_1$, then $(a+b)c = -(a+b) = -b+(-a) = -a+(-b) = ac+bc$. Now assume $c \in S_2$. If $a, b \in H$, then $a + b \in H$ and $(a + b)c = 0 = 0 + 0 = ac + bc$. If $a \in H$ and $b \notin H$, then $a + b \notin H$ and $(a + b)c = q = 0 + q = ac + bc$. If $a \notin H$ and $b \in H$, then $a + b \notin H$ and $(a + b)c = q = q + 0 = ac + bc$. If $a, b \notin H$, then $a + b \in H$ and $(a + b)c = 0 = q + q = ac + bc$. Thus, the right distributive law is satisfied.

Assume 1 is the multiplicative identity of N. Assume $1 \in S_1$ and let $x \in H$. Then $x = 1x = 0$ and $H = \{0\}$. Since H is a subgroup of G of index 2, it follows that $|G| = 2$. Since S_1, $-S_1$, S_2, and H are nonempty and mutually exclusive, G cannot have only two elements, and we have a contradiction. So assume $1 \in -S_1 \cup S_2 \cup H$. Let $x \in S_1$. Then $x = 1x = 1 \in S_1$, a contradiction. Hence, N does not have a multiplicative identity. □

The following examples show that groups satisfying the hypotheses of Theorem 3.1 exist.

Example 3.2. *Let $G = \mathbb{Z}_{2n}$ where n is odd, $H = \{0, 2, 4, \ldots, 2n - 2\}$, $S_1 = \{1, 3, 5, \ldots, n - 2\}$, and $q = n \in S_2 = \{n\}$. Then $-S_1 = \{n + 2, n + 4, \ldots, 2n - 1\}$ and $S_2 = -S_2$.*

Example 3.3. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $H = \{0\} \times \mathbb{Z}_4$, $S_1 = \{(1, 1)\}$, and $q = (1, 0) \in S_2 = \{(1, 0), (1, 2)\}$. Then $-S_1 = \{(1, 3)\}$ and $S_2 = -S_2$.*

We conclude this section with a characterization of the center and generalized center of the modified complemented Malone nearring.

Theorem 3.4. *Let N be the nearring described in Theorem 3.1. Then $C(N) = N_d = \{0, q\}$, $GC(N) = N$, and $C(N)$ is a subnearring of N.*

Proof. First, we show that $N_d \subseteq \{0, q\}$. Let $d \in N_d$. For $x \in S_1$ and $y \in S_2$, since $x, y \notin H$, then $x + y \in H$. If $d \in H$, then $0 = d(x + y) = dx + dy = d + 0 = d$ and $d = 0$. If $d \notin H$, then $0 = d(x + y) = dx + dy = d + q$ and $d = -q = q$. So $d \in \{0, q\}$ and $N_d \subseteq \{0, q\}$.

Let $x \in N$. We know $0x = 0 = x0$, so that $0 \in C(N)$. If $x \in H$, then $xq = 0 = qx$. If $x \notin H$, then $xq = q = qx$. Thus, $q \in C(N)$. We conclude that $\{0, q\} \subseteq C(N)$. This yields $C(N) \subseteq N_d \subseteq \{0, q\} \subseteq C(N)$. So $C(N) = N_d = \{0, q\}$. The rest of the theorem follows. □

4. A Final Construction

Two of the constructions in [4], including the TS nearrings, used sets T and S to define the nearring multiplication where the set S was partitioned into smaller sets. In the following construction, we partition the set T instead of the set S.

Theorem 4.1. *Let $(G, +)$ be a group, not necessarily abelian. Let T_0 be a proper normal subgroup of G with $G/T_0 \cong \mathbb{Z}_{n+1}$, T_1 be a generator of G/T_0, and $T_2, \ldots, T_n$ be the remaining nonzero cosets of G determined by T_0 with $T_j = jT_1$ for $j = 1, 2, \ldots, n$. Furthermore, let $\emptyset \neq I \subseteq \{1, 2, \ldots, n\}$ and $\cup_{i \in I} T_i = S \subseteq \cup_{i=1}^{n} T_i = T$. Let $0 \neq q \in T_0 \cup T_1$ such that $|q| = n + 1$.*

Define a multiplication on G by

$$a \cdot b = \begin{cases} iq, & \text{if } a \in T_i, b \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Then $N = (G, +, \cdot)$ is a right, zero-symmetric nearring.

Proof. Since $(G, +)$ is a group, we only need to check properties involving multiplication. Since $0 \in T_0$, then $a0 = 0a = 0$ for all $a \in N$, and N is zero-symmetric. We note that since $T_j = jT_1$ for $j = 1, 2, \ldots, n$, and T_1 is a generator of $G/T_0 \cong \mathbb{Z}_{n+1}$, then addition of cosets is modulo $n + 1$, i.e., $T_i + T_j = T_{(i+j)\text{mod}(n+1)}$. In particular, if $q \in T_0$, then $iq \in T_0$, and if $q \in T_1$, then $iq \in T_i$.

To show associativity of multiplication, we consider two cases. If $q \in T_0$, then for all $a, b, c \in N$, we have $ab, bc \in T_0$. It follows that $(ab)c = 0 = a(bc)$. Now assume $q \in T_1$. If $c \notin S$, $a \in T_0$, or $b \in T_0$, then $(ab)c = 0 = a(bc)$. Thus, we assume that $c \in S$ and $a, b \in T$. If $a \in T_j$ and $b \in T_k \subseteq S$, then $(ab)c = (jq)c = jq = a(kq) = a(bc)$. If $a \in T_j$ and $b \in T_k \not\subseteq S$, then $(ab)c = 0c = 0 = a(kq) = a(bc)$. Thus, multiplication is associative.

Now we show that the right distributive law holds. If $c \notin S$, then $(a+b)c = 0 = 0+0 = ac+bc$. So assume $c \in S$. If $a, b \in T_0$, then $a+b \in T_0$ and $(a+b)c = 0 = 0+0 = ac+bc$. If $a \in T_0$ and $b \in T_j \subseteq T$, then $a+b \in T_j$ and $(a+b)c = jq = 0+jq = ac+bc$. If $a \in T_j \subseteq T$ and $b \in T_0$, then $a+b \in T_j$ and $(a+b)c = jq = jq+0 = ac+bc$. If $a \in T_i \subseteq T$ and $b \in T_j \subseteq T$, then for $k = (i+j)\mathrm{mod}(n+1)$, $a+b \in T_k$ and $(a+b)c = kq = iq+jq = ac+bc$ since $|q| = n+1$. □

The next theorem shows that this nearring rarely has a multiplicative identity.

Theorem 4.2. *For the nearring N defined in Theorem 4.1, N has a multiplicative identity if and only if $N \cong \mathbb{Z}_2$.*

Proof. Assume N has multiplicative identity, 1. If $1 \in T_0$, then for $0 \neq a \in N$, $a = 1a = 0$, a contradiction. So $1 \notin T_0$ and $1 \in T$. If $1 \notin S$, then $1 = 1 \cdot 1 = 0 \in T_0$, a contradiction. Hence, $1 \in S$. Let $x \notin S$. Then $0 = 1x = x$. So $S = T$, $T_0 = \{0\}$, and $G \cong \mathbb{Z}_{n+1}$. It follows that $T_1 = \{q\}, T_2 = \{2q\}, \dots, T_n = \{nq\}$. Since $1 \in S = T$, then $1 = kq \in T_k$ for some $1 \leq k \leq n$.

Assume $n \geq 2$. Let $x \in T_j \not\subseteq T_0 \cup T_k$. Then $1 = kq = 1x = x \in T_j$, a contradiction. We conclude that $n = 1$ and $G \cong \mathbb{Z}_2$. Letting $T_0 = \{0\}$, $T_1 = \{1\}$, $I = \{1\}$, and $q = 1$ yields $N \cong \mathbb{Z}_2$. The converse is clear. □

To prove our characterization theorem, we first need a few lemmas.

Lemma 4.3. *Let N be the nearring defined in Theorem 4.1, and let $d \in N_d$. Suppose that any one of the following three conditions are satisfied:*
(1) There exist $x, y \in S$ such that $x+y \in S$;
(2) There exist $x \in S$ and $y \notin S$ such that $x+y \notin S$ or $y+x \notin S$;
(3) There exist $x, y \notin S$ such that $x+y \in S$.
Then $d \in T_0$.

Proof. For all three cases, assume $d \in T_i$.

(1) If $x, y \in S$ and $x+y \in S$, then $iq = d(x+y) = dx+dy = iq+iq$. Thus, $iq = 0$ and $i = 0$. So $d \in T_0$.

(2) If $x \in S$, $y \notin S$ and $x+y \notin S$, then $0 = d(x+y) = dx+dy = iq+0 = iq$. Thus, $i = 0$ and $d \in T_0$. A similar proof works for $y+x \notin S$.

(3) If $x, y \notin S$ and $x+y \in S$, then $iq = d(x+y) = dx+dy = 0+0 = 0$. Thus, $i = 0$ and $d \in T_0$. □

Lemma 4.4. *Let N be the nearring defined in Theorem 4.1. If there exists $d \in N_d$ such that $d \notin T_0$, then n is odd and $I = \{1, 3, 5, \dots, n\}$.*

Proof. Assume there exists $d \in N_d$ with $d \notin T_0$. Suppose $T_1 \subseteq T \backslash S$. Note that if $T_1 + T_1 = T_0$, then $S = \emptyset$, a contradiction. So $T_1 + T_1 \neq T_0$. If

$T_1 + T_1 = T_2 \subseteq S$, then by part (3) of Lemma 4.3, $d \in T_0$, a contradiction. So $T_2 \subseteq T \backslash S$. If $T_2 + T_1 = T_3 \subseteq S$, then $d \in T_0$ by part (3) of Lemma 4.3, a contradiction. Continuing in this manner yields $I = \emptyset$, a contradiction. Thus, $T_1 \subseteq S$ and $1 \in I$.

If $T_1 + T_1 = T_0$, we are done. So assume $T_1 + T_1 \neq T_0$. If $T_1 + T_1 = T_2 \subseteq S$, then by part (1) of Lemma 4.3, $d \in T_0$, a contradiction. So $T_2 \subseteq T \backslash S$. If $T_2 + T_1 = T_3 \subseteq N \backslash S$, then by part (2) of Lemma 4.3, $d \in T_0$, a contradiction. So $T_3 \subseteq S$ and $3 \in I$.

Assume n is even. Continuing in the above manner yields $I = \{1, 3, 5, \ldots, n-1\}$. So $T_1 \subseteq S$ and $T_n \subseteq T \backslash S$. Thus for $x \in T_1$ and $y \in T_n$, $x + y \in T_0$ and $x + y \notin S$. By part (2) of Lemma 4.3, $d \in T_0$, a contradiction. It follows that n is odd. Continuing the construction above yields $I = \{1, 3, 5, \ldots, n\}$. □

We are now ready to state and prove our characterization theorem.

Theorem 4.5. *Let N be the nearring defined in Theorem 4.1.*
(1) If $n = 1$, then $S = T$ and $C(N) = N_d = GC(N) = N$.
(2) If n is odd, $n \geq 3$, and $I = \{1, 3, 5, \ldots, n\}$, then $C(N) = T_0$ and $N_d = T_0 \cup T_k$, where $k = \frac{1}{2}(n+1)$. Furthermore, if k is odd, then $GC(N) = T_0 \cup T_k$. If k is even, then $GC(N) = N \backslash S = T_0 \cup T_2 \cup \cdots \cup T_{n-1}$.
(3) Otherwise, $C(N) = N_d = T_0 \subsetneq GC(N) = N$.

In all cases, $C(N)$ is a subnearring of N.

Proof. We first note that in all cases, if $c \in T_0$ and $x \in N$, then $cx = 0 = xc$. So $c \in C(N)$ and $T_0 \subseteq C(N)$.

To finish the proof of (1), assume $n = 1$. Then $I = \{1\}$ and $S = T_1 = T$. Assume $c \in T_1$. If $a \in T_0$, then $ca = 0 = ac$. If $a \in T_1$, then $ca = q = ac$. Thus, $c \in C(N)$ and $T_1 \subseteq C(N)$. We conclude that $N = T_0 \cup T_1 \subseteq C(N) \subseteq N_d \subseteq N$. Since $C(N) \subseteq GC(N) \subseteq N$, the result now follows.

To prove (2), let $3 \leq n$ be odd, $I = \{1, 3, \ldots, n\}$, and $k = \frac{1}{2}(n+1)$. From the comment above, we know $T_0 \subseteq C(N)$. Let $c \in C(N)$, say $c \in T_i$. Assume $c \in S$. Then for $x \in T_1 \subseteq S$, $iq = cx = xc = q$. So $i = 1$. For $y \in T_3 \subseteq S$, $iq = cy = yc = 3q$, and $i = 3$, a contradiction. So $c \notin S$. Now let $z \in T_1 \subseteq S$. Then $iq = cz = zc = 0$. So $i = 0$ and $c \in T_0$. It follows that $C(N) \subseteq T_0$ and $C(N) = T_0$.

To determine the distributive elements, we know $T_0 \subseteq C(N) \subseteq N_d$. Let $a \in T_k$. Since $I = \{1, 3, \ldots, n\}$, it follows that S is the union of all cosets with odd subscripts and $N \backslash S$ is the union of all cosets with even subscripts, including T_0. Let $x, y \in N$. There are four cases to consider. If $x, y \in S$, then $x + y \notin S$ and $a(x + y) = 0 = (n + 1)q = kq + kq = ax + ay$. If $x \in S$ and $y \notin S$, then $x + y \in S$ and $a(x + y) = kq = kq + 0 = ax + ay$. If $x \notin S$ and $y \in S$, then $x + y \in S$ and $a(x + y) = kq = 0 + kq = ax + ay$. If

$x, y \notin S$, then $x + y \notin S$ and $a(x + y) = 0 = 0 + 0 = ax + ay$. So $a \in N_d$ and $T_k \subseteq N_d$. Thus, $T_0 \cup T_k \subseteq N_d$.

Now we show containment in the other direction. Let $d \in N_d$. If $d \in T_0$, then we are done. So assume $d \in T_i \neq T_0$. Let $x \in T_1 \subseteq S$. Then $x + x \in T_2 \subseteq N \backslash S$ and $0 = d(x + x) = dx + dx = iq + iq = (2i)q$. Thus $(n + 1)|(2i)$. Since $1 \leq i \leq n$, then $2i \leq 2n < 2n + 2 = 2(n + 1)$. It follows that $n + 1 = 2i$ and $i = \frac{1}{2}(n + 1) = k$. Thus, $d \in T_k$. Hence, $N_d \subseteq T_0 \cup T_k$ and $N_d = T_0 \cup T_k$.

We now consider $GC(N)$. We have already established $T_0 \subseteq C(N) \subseteq GC(N)$.

Assume k is odd. Consider $a \in T_k \subseteq S$. Let $y \in N_d = T_0 \cup T_k$. If $y \in T_0$, then $ay = 0 = ya$. If $y \in T_k \subseteq S$, then $ay = kq = ya$. Thus, $a \in GC(N)$, $T_k \subseteq GC(N)$, and $T_0 \cup T_k \subseteq GC(N)$. For containment in the other direction, let $a \in GC(N)$. Assume $a \in T_i$. Let $x \in T_k \subseteq N_d$. So $x \in S$. If $a \notin S$, then $iq = ax = xa = 0$. So $i = 0$ and $a \in T_0$. If $a \in S$, then $iq = ax = xa = kq$. So $i = k$ and $a \in T_k$. It follows that $GC(N) \subseteq T_0 \cup T_k$, hence equality.

To complete the proof of (2), assume k is even. Let $a \in N \backslash S$, say, $a \in T_i$ where i is even. Let $y \in N_d = T_0 \cup T_k$. Then $y \notin S$ and $ay = 0 = ya$. So $a \in GC(N)$ and $N \backslash S \subseteq GC(N)$. For the reverse containment, let $a \in GC(N)$. Let $x \in T_k \subseteq N_d$. So $x \notin S$. If $a \in S$, then $0 = ax = xa = kq$, a contradiction. Hence, $a \in N \backslash S$. It follows that $GC(N) \subseteq N \backslash S$, hence equality.

For part (3), assume either n is even, or n is odd with $n \geq 3$ and $I \neq \{1, 3, 5, \ldots, n\}$. Let $d \in N_d$. By Lemma 4.4, $d \in T_0$. So $T_0 \subseteq C(N) \subseteq N_d \subseteq T_0$. It follows that $C(N) = N_d = T_0 \subsetneq GC(N) = N$.

It is clear that $C(N)$ is a subnearring of N in case (1) since $C(N) = N$. In cases (2) and (3), since $C(N) = T_0$ and T_0 is closed under addition and multiplication, it follows that $C(N)$ is a subnearring of N. □

We conclude the paper with examples for each case of Theorem 4.5.

Example 4.6. *Let $G = \mathbb{Z}_6$ and $T_0 = \{0, 2, 4\}$. Then $G/T_0 \cong \mathbb{Z}_2$, $n = 1$, and $S = T_1 = T = \{1, 3, 5\}$. Letting $q = 3 \in T_1$ yields case (1).*

Example 4.7. *Let H be any finite group, $3 \leq n$ be a positive odd integer, and $G = \mathbb{Z}_{n+1} \times H$. Let $T_0 = \{0\} \times H$. Then $G/T_0 \cong \mathbb{Z}_{n+1}$. Letting $T_1 = (1, 0) + T_0$, $I = \{1, 3, 5, \ldots, n\}$, and $q = (1, 0) \in T_1$ yields case (2). If $n = 7$, then k is even; if $n = 9$, then k is odd.*

Example 4.8. *Let $G = \mathbb{Z}_{12}$ and $T_0 = \{0, 4, 8\}$. Then $G/T_0 \cong \mathbb{Z}_4$ and $n = 3$. Letting $T_1 = 1 + T_0$, $I = \{1, 2\}$, and $q = 9$ yields case (3).*

References

[1] E. Aichinger, F. Binder, J. Ecker, R. Eggetsberger, P. Mayr, and C. Nöbauer, *SONATA - System of Near-Rings and Their Applications*, Package for the group theory system GAP4, Division of Algebra, Johannes Kepler University, Linz, Austria, 1999.

[2] E. Aichinger and M. Farag, *On when the multiplicative center of a near-ring is a subnear-ring*, Aequationes Math., **68.1-2** (2004), 46–59.

[3] G. A. Cannon, M. Farag, and L. Kabza, *Centers and generalized centers of near-rings*, Comm. Algebra, **35.2** (2007), 443–453.

[4] G. A. Cannon, M. Farag, L. Kabza, and K. Neuerburg, *Centers and generalized centers of near-rings without identity defined via Malone-like multiplications*, to appear in Math. Pannonica.

[5] M. Farag, *A new generalization of the center of a near-ring with applications to polynomial near-rings*, Comm. Algebra, **29.6** (2001), 2377–2387.

[6] J. D. P. Meldrum, *Near-Rings and Their Links with Groups*, Research Notes in Math., 134, Pitman Publ. Co., Boston, 1985.

[7] G. Pilz, *Near-Rings*, North-Holland/American Elsevier, Amsterdam, 1983.

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Department of Mathematics, Southeastern Louisiana University, Hammond, LA 70402

E-mail address: acannon@southeastern.edu

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849

E-mail address: vjg0004@auburn.edu

Department of Mathematics, Southeastern Louisiana University, Hammond, LA 70402

E-mail address: Bailey.Hall@southeastern.edu

Department of Mathematics and Statistics, Boston University, Boston, MA 02215

E-mail address: tdtriche@bu.edu