INTERSECTION THEOREMS FOR CLOSED CONVEX SETS AND APPLICATIONS

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Abstract. A number of landmark existence theorems of nonlinear functional analysis follow in a simple and direct way from the basic separation of convex closed sets in finite dimension via elementary versions of the Knaster-Kuratowski-Mazurkiewicz principle - which we extend to arbitrary topological vector spaces - and a coincidence property for so-called von Neumann relations. The method avoids the use of deeper results of topological essence such as the Brouwer Fixed Point Theorem or the Sperner's Lemma and underlines the crucial role played by convexity. It turns out that the convex KKM Principle is equivalent to the Hahn-Banach Theorem, the Markov-Kakutani Fixed Point Theorem, and the Sion-von Neumann Minimax Principle.

1. Introduction

The aim of this expository paper is to show that a number of landmark results of nonlinear functional analysis can be quickly obtained from a particular version of the KKM Principle at little cost. This *Elementary KKM Principle* is due to A. Granas and M. Lassonde in the framework of super-reflexive Banach spaces [10]. It is extended to arbitrary topological vector spaces, under a more general compactness hypothesis, with a simpler proof based on the separation of closed convex subsets in a Euclidean space (a result usually discussed in a first course of continuous optimization) and an intersection theorem of V. L. Klee [17]. A similar approach is followed to formulate a coincidence theorem for so-called *von Neumann relations*.

The methods outlined here allow for a shorter and simpler alternative treatment of existence results of functional analysis that avoids involved and deeper principles that require sophistication and investment in time. The KKM Principle is a striking example of such fundamental results. Indeed, using the Sperner lemma as a starting point, three of the greatest topologists of all times, Polish academician S. Mazurkiewicz and two of his former doctoral students, B. Knaster and K. Kuratowski published in 1929 the celebrated KKM Lemma: a remarkable intersection theorem for closed covers of a Euclidean simplex [18]. They used the KKM Lemma to provide a combinatorial proof of the Brouwer Fixed Point Theorem (the two results being, in fact, equivalent). In 1961, Ky Fan extended the KKM Lemma to vector spaces of arbitrary dimensions in what became known as the KKM Principle [12]. The KKM Principle inspired countless mathematicians, yielding a formidable body of work in nonlinear and convex analysis; a production known today as the KKM Theory. The reader is referred to Dugundji-Granas [7], Park [20] and Yuan [22] for surveys of results, methods, and applications of the KKM Theory.

The particular version of the KKM Principle discussed here, which we call the *convex KKM Principle*, is more than sufficient to prove in a direct and economical way, such fundamental results as the Stampacchia Theorem on variational inequalities, the Mazur-Schauder Theorem on the minimization of lower semicontinuous quasiconvex and coercive functionals, and the Markov-Kakutani Fixed Point Theorem for commuting families of affine transformations (see e.g., Brézis [6]). It is well-known, since Kakutani [16], that the Hahn-Banach Theorem can be derived from the Markov-Kakutani Fixed Point Theorem. Thus, the equivalence between the Hahn-Banach Theorem, Klee's intersection theorem, the convex KKM Principle, and the Markov-Kakutani Fixed Point Theorem is thus established.

2. Preliminaries

The fundamental tool for our proof of the convex KKM Theorem is the separation of a point and a closed convex set in a finite dimensional space. For the sake of completeness, we include the basic separation properties in finite dimensions with the simplest of proofs (see e.g., Magill and Quinzii [19]).

Lemma 1. *Let C be a non-empty closed convex subset of $\mathbb{R}^n$ and let $x \notin C$. Denote by $y = P_C(x)$ the projection of x onto C. Then the hyperplane H_C^x, orthogonal to $u = x - y$, passing through y strictly separates x and C, namely*

$$\langle u, z\rangle \leq \langle u, y\rangle < \langle u, x\rangle, \text{ for all } z \in C.$$

Proof. Since C is closed and convex, the projection $y = P_C(x)$ of x onto C is unique. Define, for any given $z \in C$, a functional $\varphi_z : [0,1] \longrightarrow \mathbb{R}$ by

$$\varphi_z(t) := \|x - (tz + (1-t)y\|^2.$$

As y is closest to x, $\varphi_z(t)$ achieves its minimum on [0, 1] at $t = 0$, thus $\varphi_z'(0) \geq 0$. Since $\varphi_z'(t) = 2t\|y - z\|^2 + 2\langle x - y, y - z\rangle$, it follows $\varphi_z'(0) = 2\langle x - y, y - z\rangle = 2\langle u, y - z\rangle \geq 0$, i.e., $\langle u, z\rangle \leq \langle u, y\rangle$. On the other hand, $0 < \|x-y\|^2 = \langle u, x-y\rangle = \langle u, x\rangle - \langle u, y\rangle$. Thus, $\langle u, z\rangle \leq \langle u, y\rangle < \langle u, x\rangle$. □

Proposition 2. *Let K and C be disjoint convex subsets of $\mathbb{R}^n$ with K compact and C closed. Then, C and K are strictly separated by a hyperplane H, i.e.,*

$$\textit{there exists } u \in \mathbb{R}^n,\ u \neq 0,\ \textit{with } \sup_{x \in C} \langle u, x \rangle < \min_{x' \in K} \langle u, x' \rangle.$$

Proof. Since C is closed and K is compact, the set $C - K := \{y \in \mathbb{R}^n : y = x - x', x \in C, x' \in K\}$ is also closed. Moreover, it is convex as the difference of convex sets. Since $C \cap K = \emptyset$, then $0 \notin C - K$. Lemma 1 applies, yielding for $u = 0 - P_{C-K}(0)$, the inequalities: $\langle u, z \rangle \leq \langle u, -u \rangle < \langle u, 0 \rangle = 0$, for all $z \in C - K$. Thus, as $z = x - x', x \in C, x' \in K$,

$$\langle u, x \rangle \leq \langle u, x' \rangle - \|u\|^2 < \langle u, x' \rangle, \text{ for all } x \in C, \text{ for all } x' \in K.$$

$\square$

A refinement of a fundamental intersection theorem of V. L. Klee for families of closed convex subsets of $\mathbb{R}^n$ (see Klee [17] and Berge [5]) plays a crucial role in our proof. We provide here a simple proof based on Proposition 2.

Topological vector spaces (t.v.s. for short), as well as topological spaces, are assumed to be Hausdorff (T_2). Vector spaces are assumed real (or complex) and the convex hull of a subset A of a vector space is denoted by $conv(A)$.

Proposition 3. *([11]) Let $C_1, \ldots, C_n$, be non-empty closed convex sets in a t.v.s. E such that:*

(i) $C = \bigcup_{i=1}^n C_i$ *is convex, and*
(ii) *each k of them, $1 \leq k < n$, have a common point.*

Then $\bigcap_{i=1}^n C_i \neq \emptyset$.

Proof. The proof goes along the lines of Klee's proof [17]. One may assume with no loss of generality that the sets $C_i, i = 1, \ldots, n$, are compact convex subsets of a finite dimensional space. Indeed, one could consider the convex finite polytope $\hat{C} := Conv(\{y_j : j = 1, \ldots, n\})$, where, for each $j = 1, \ldots, n$, the points $y_j \in \bigcap_{i=1, i \neq j}^n C_i$ are provided by (ii), and define $\hat{C}_i := C_i \cap \hat{C}$. Clearly, all the sets $\hat{C}_1, \ldots, \hat{C}_n, \hat{C} = \bigcup_{i=1}^n \hat{C}_i$ are compact convex sets in a finite dimensional subspace of E and $\bigcap_{i=1}^n C_i \neq \emptyset \Longleftrightarrow \bigcap_{i=1}^n \hat{C}_i \neq \emptyset$.

If $n = 1$, the thesis clearly holds. Assume, for a contradiction that for $n \geq 2$, $\bigcap_{i=1}^n C_i = \emptyset$ and let us show that (i) must fail if (ii) holds true. The proof is by induction on n.

If $n = 2$, (ii) asserts that both C_1 and C_2 are non-empty and, while they are disjoint, their union $C = C_1 \cup C_2$ cannot be convex and thus (i) fails.

Suppose that for $n = k-1$, it holds $(\bigcap_{i=1}^{k-1} C_i = \emptyset$ and $\bigcap_{i=1,i\neq j}^{k-1} C_i \neq \emptyset) \Longrightarrow \bigcup_{i=1}^{k-1} C_i$ is not convex.

Let $n = k$, and let $\{C_i\}_{i=1}^{k}$ be a collection of compact convex sets such that $C_k \cap \bigcap_{i=1}^{k-1} C_i = \emptyset$ and for all $j = 1,\ldots,k$, $\bigcap_{i=1,i\neq j}^{k} C_i \neq \emptyset$. By Proposition 2, the disjoint compact convex sets C_k and $\bigcap_{i=1}^{k-1} C_i$ can be strictly separated by a hyperplane H. Putting, for each $i = 1,\ldots,k$, $C_i' := H \cap C_i$, it follows that C_k' and $\bigcap_{i=1}^{k-1} C_i'$ are empty. Moreover, for a given arbitrarily chosen $j_0 \in \{1,\ldots,k-1\}$, let $y_0 \in \bigcap_{i=1,i\neq j_0}^{k} C_i$, thus $y_0 \in C_k$, and let $y_k \in \bigcap_{i=1}^{k-1} C_i$ be arbitrarily chosen. Clearly, the points y_0 and y_k are both in the larger convex set $\bigcap_{i=1,i\neq j_0}^{k-1} C_i$ and are also strictly separated by H. The intersection $\bar{z}$ of the line segment $[y_0, y_k]$ with H belongs to $\bigcap_{i=1,i\neq j_0}^{k-1} C_i \cap H$. Since j_0 is arbitrary, hypothesis (ii) is verified for the collection $\{C_i'\}_{i=1}^{k-1}$ and $\bigcap_{i=1}^{k-1} C_i' = \emptyset$. By the induction hypothesis, $\bigcup_{i=1}^{k-1} C_i' = \bigcup_{i=1}^{k-1}(C_i \cap H)$ is not convex. Since $H \cap C_k = \emptyset$, it follows that $\bigcup_{i=1}^{k}(C_i \cap H) = \bigcup_{i=1}^{k-1}(C_i \cap H)$ is not convex and the proof is complete. □

Remark 4. *Proposition 3 is due to A. Ghouila-Houri [11] and slightly extends the following result of V. L. Klee (see also C. Berge [6]).*

Theorem 5. *(Klee's Theorem) [17]. Let C and $C_1,\ldots,C_n$ be closed convex sets in a Euclidean space satisfying*

(i) $C \subseteq \bigcup_{i=1}^{n} C_i$
(ii) $C \cap \bigcap_{i=1,i\neq j}^{n} C_i \neq \emptyset$ *for any* $j = 1,2,\ldots,n$.

Then $C \cap \bigcap_{i=1}^{n} C_i \neq \emptyset$.

This can be restated: $(C \cap \bigcap_{i=1,i\neq j}^{n} C_i \neq \emptyset$ *and* $C \cap \bigcap_{i=1}^{n} C_i = \emptyset) \Longrightarrow C \nsubseteq \bigcup_{i=1}^{n} C_i$.

3. The Convex KKM Theorem

We use the following terminology of Dugundji-Granas (see [7]).

Definition 6. *Given an arbitrary subset X be of a vector space E, a set-valued map $\Gamma : X \longrightarrow 2^E$ is said to be a KKM map if for every finite subset $\{x_1,\ldots,x_n\} \subseteq X$ it holds:*

$$conv(\{x_1,\ldots,x_n\}) \subset \bigcup_{i=1}^{n} \Gamma(x_i).$$

Theorem 7. *(Convex KKM Theorem) Let E be a t.v.s., $\emptyset \neq X \subseteq Y \subseteq E$ with Y convex. If $\Gamma : X \longrightarrow 2^Y$ is a set-valued map verifying:*

(i) Γ *is a KKM map;*
(ii) *all values of* Γ *are non-empty, closed and convex.*

Then, the family $\{\Gamma(x)\}_{x\in X}$ *has the finite intersection property.*
If in addition, there exists a non-empty subset X_0 *of* X *contained in a convex compact subset* D *of* Y *such that* $\bigcap_{x\in X_0}\Gamma(x)$ *is compact, then* $\bigcap_{x\in X}\Gamma(x)\neq\varnothing$.

Proof. We prove that Proposition 3 is equivalent to Theorem 7.

($\Longrightarrow$) Let $\Gamma : X \longrightarrow 2^Y$ be a KKM map with closed convex values. We show by induction on n that $conv(\{x_1,\dots,x_n\})\cap\bigcap_{i=1}^n\Gamma(x_i)\neq\emptyset$, for any finite subset $\{x_1,\dots,x_n\}$ of X.

When $n=1$, $x_1 = conv(\{x_1\})\subset\Gamma(x_1)$.

Assume that the conclusion holds true for any set with $n=k$ elements, and let $n=k+1$. Put $C = conv(\{x_1,\dots,x_n\})$ and $C_i=\Gamma(x_i)\cap C$. Since Γ is KKM, $C\subseteq\bigcup_{i=1}^n\Gamma(x_i)$ which implies $C=\bigcup_{i=1}^n(\Gamma(x_i)\cap C) = \bigcup_{i=1}^n C_i$, a convex set. By the induction hypothesis, for each i, we have $conv(\{x_1,\dots,\hat{x}_i,\dots,x_n\})\cap\bigcap_{j=1,j\neq i}^n\Gamma(x_j)\neq\emptyset$. Proposition 3 implies that $\bigcap_{i=1}^n(\Gamma(x_i)\cap C)\neq\emptyset$, i.e., $\bigcap_{i=1}^n\Gamma(x_i)\neq\emptyset$.

($\Longleftarrow$) Assume $C_1,\dots,C_n, C=\bigcup_{i=1}^n C$ are closed convex sets in a topological vector space satisfying hypotheses (i) and (ii) of Proposition 3 above.

For each j, let $x_j\in\bigcap_{i=1,i\neq j}^n C_i$ and consider $X=\{x_j\}_{j=1}^n$. The set C being convex, $conv(X)\subseteq C$ and for all j,i with $j\neq i$, $x_j\in C_i$, which implies that $A_i = conv(\{x_j\}_{j=1,j\neq i}^n)\subset C_i$. Define $\Gamma : X\longrightarrow 2^C$ by $\Gamma(x_i) := C_i$ for each $i=1,\dots,n$. The values of Γ are clearly closed and convex. Also, $conv(X)\subseteq C=\bigcup_{i=1}^n(C_i\cap C)=\bigcup_{i=1}^n\Gamma(x_i)$, and for each $\{x_{i_1},\dots,x_{i_k}\}\subset X$, we have $conv(\{x_{i_1},\dots,x_{i_k}\})\subset A_{i_j}\subset C_{i_j}=\Gamma(x_{i_j})$ for some $j\neq 1,\dots,k$. Hence, $conv(\{x_{i_1},\dots,x_{i_k}\})\subset\bigcup_{j=1}^k\Gamma(x_{i_j})$, i.e., Γ is a KKM map. By the convex KKM Theorem, $\bigcap_{i=1}^n\Gamma(x_i)\neq\emptyset$, thus $\bigcap_{i=1}^n C_i\neq\emptyset$.

Assuming for a moment that $\bigcap_{x\in X}\Gamma(x)$ is contained in a compact subset K of Y, then the conclusion $\bigcap_{x\in X}\Gamma(x)\neq\varnothing$ would follow at once from the characterization of compactness in terms of families of closed subsets having the finite intersection property.

Observe now that the restriction/compression map $\Gamma_0 : X_0\longrightarrow 2^D$ defined by $\Gamma_0(x) := \Gamma(x)\cap D, x\in X_0$, has compact convex values and is also a KKM map. Indeed, for any subset $\{x_1,\dots,x_n\}\subseteq X_0$, $conv(\{x_1,\dots,x_n\})\subset(\bigcup_{i=1}^n\Gamma(x_i))\cap D=\bigcup_{i=1}^n\Gamma_0(x_i)$. Therefore, $\bigcap_{x\in X_0}\Gamma(x)\supseteq\bigcap_{x\in X_0}\Gamma_0(x)\neq\varnothing$. The conclusion follows immediately from the fact that $\bigcap_{x\in X}\Gamma(x)\subseteq\bigcap_{x\in X_0}\Gamma(x)$ is compact and non-empty. $\square$

Remark 8.

(i) *Theorem 7 is an extension to topological vector spaces of the elementary KKM Theorem of Granas-Lassonde, stated in the context of super-reflexive Banach spaces [10].*
(ii) *Theorem 7 obviously follows from the KKM Principle of Ky Fan [12] where the values of Γ are not assumed to be convex. However, the latter requires much more involved analytical or topological results. Indeed, the Ky Fan KKM Principle is equivalent to Sperner's Lemma, to the Brouwer Fixed Point Theorem, and to the Browder-Ky Fan Fixed Point Theorem (see e.g., [1, 2, 3]).*
(iii) *In this generality, the compactness condition in the KKM Principle is due to Ky Fan [14]. It obviously extends the earlier compactness conditions: Y is also compact, or all values of Γ are compact, or a single value $\Gamma(x_0)$ is compact, or $\bigcap_{i=1}^n \Gamma(x_i)$ is compact for some finite subset $\{x_1, \ldots, x_n)$ of X.*

Naturally, the convex KKM Theorem can be expressed as an equivalent fixed point property for what we call a *von Neumann relation*. Given a subset A of a cartesian product of two sets $X \times Y$, denote by $A(x)$ and $A^{-1}(y)$ the respective sections $\{y \in Y : (x, y) \in A\}$ and $\{x \in X : (x, y) \in A\}$; denote by A^{-1} the subset $\{(y, x) : (x, y) \in A\}$.

Definition 9. *A von Neumann relation is a subset A of a cartesian product $X \times Y$, where X and Y are subsets of topological vector spaces, satisfying*
(i) *for every $x \in X$, the section $A(x)$ is convex and non-empty;*
(ii) *for every $y \in Y$, the section $A^{-1}(y)$ is open in X and $X \setminus A^{-1}(y)$ is convex.*

Denote by $\mathcal{N}(X, Y)$ the class of von Neumann relations in $X \times Y$ and by $\mathcal{N}^{-1}(X, Y) := \{A : X \longrightarrow 2^Y : A^{-1} \in \mathcal{N}(Y, X)\}$.

Note that von Neumann relations are particular cases of F^*-maps (*applications de Ky Fan*) introduced in [3].

Theorem 10. *(Fixed Point for $\mathcal{N}-$maps) Let E be a t.v.s., $\emptyset \neq Y \subseteq X \subseteq E$ with X convex, and let $A \in \mathcal{N}(X, Y)$. If there exist a compact subset K of X and a compact convex subset D of Y such that for every $x \in X \setminus K$, $A(x) \cap D \neq \emptyset$, then A has a fixed point, i.e., $(\hat{x}, \hat{x}) \in A$ for some $\hat{x} \in X$.*

Proof. Define $\Gamma : Y \longrightarrow 2^X$ as $\Gamma(y) := Y \setminus A^{-1}(y), y \in Y$. Clearly, Γ has closed and convex values. Also, obviously, $A(x) = Y \setminus \Gamma^{-1}(x)$, for any $x \in X$.

One readily verifies that the compactness condition in Theorem 10 is equivalent to the compactness condition in Theorem 7. Indeed, ($x \notin K \Longrightarrow$

there exists $y \in A(x) \cap D) \iff ((\text{ for all } y \in D, y \notin A(x)) \implies x \in K) \iff (\bigcap_{y\in D}\Gamma(y) \subseteq K)$. The intersection $\bigcap_{y\in D}\Gamma(y)$ being closed in K is compact. For any subset Y_0 of D, it also holds $\bigcap_{y\in Y_0}\Gamma(y)$ is a compact subset of K.

The fact that all sections $A(x), x \in X$, are non-empty rules out the thesis of Theorem 7 (indeed, $(A(x) \neq \emptyset$, for all $x \in X) \iff \bigcap_{y\in Y}\Gamma(y) = \varnothing)$. Therefore Γ cannot be a KKM map, i.e., there exist $\{y_1,\ldots,y_n\} \subseteq Y$ and $\hat{y} \in conv(\{y_1,\ldots,y_n\})$ with $\hat{y} \notin \bigcup_{i=1}^n \Gamma(y_i)$, which (by DeMorgan's law) is equivalent to $\hat{y} \in \bigcap_{i=1}^n A^{-1}(y_i) \iff \{y_1,\ldots,y_n\} \subseteq A(\hat{y})$. Since $A(\hat{y})$ is convex, $\hat{y} \in A(\hat{y})$ and the proof is complete. $\square$

Remark 11.

(i) *The proof of Theorem 10 clearly establishes its equivalence with Theorem 7.*

(ii) *Theorem 10 is a particular instance of the Browder-Ky Fan fixed point theorem (where the convexity of $X \setminus A^{-1}(y)$ in Definition 9 is dispensed with; see e.g., [2, 3, 4]).*

(iii) *Note that if X is compact, the compactness condition in Theorem 10 is vacuously satisfied with $K = X$. To the best of our knowledge, in this generality and in the context of the Browder-Ky Fan Fixed Point Theorem, this condition was first introduced in [2, 3] in 1982. It builds on the so-called Karamardian coercivity condition for complementarity problems (early seventies), taken up in 1977 by Allen in the context of fixed point theorems for set-valued maps (case where $K = C$); see [2, 3, 9] for references and details.*

We end this section with a coincidence theorem between $\mathcal{N}$ and $\mathcal{N}^{-1}$ maps with a direct proof based on Proposition 3. We shall make use of a well-known selection property enjoyed by F^*-maps of [2], thus by $\mathcal{N}-$maps (see [2, 3]).

Lemma 12. *Let $A \in \mathcal{N}(X,Y)$ with Y convex. For any compact subset K of X, there exist a continuous (single-valued) mapping $s : K \longrightarrow Y$ and a convex compact finite polytope $P \subseteq Y$ such that $s(x) \in A(x) \cap P$ for all $x \in K$.*

Proof. Since for all $x \in X, A(x) \neq \emptyset$, then $X = \bigcup_{y\in Y} A^{-1}(y)$, a union of open subsets of X. By compactness, $K \subseteq \bigcup_{i=1}^n A^{-1}(y_i)$ for some finite subset $\{y_1,\ldots,y_n\} \subset Y$. Let $\{\lambda_i : K \longrightarrow [0,1]\}_{i=1}^n$ be a continuous partition of unity subordinated to the open cover $\{A^{-1}(y_i)\}_{i=1}^n$, and define $s : K \longrightarrow P = conv(\{y_1,\ldots,y_n\}) \subset Y$ by putting $s(x) := \sum_{i=1}^n \lambda_i(x)y_i, x \in K$. Clearly, for a given $x \in K$, $\lambda_i(x) \neq 0 \implies x \in A^{-1}(y_i) \iff y_i \in A(x)$. The section $A(x)$ being convex, the convex combination $s(x) \in A(x) \cap P$. $\square$

Theorem 13. *(Coincidence $(\mathcal{N},\mathcal{N}^{-1})$) Let X and Y be two non-empty convex subsets in topological vector spaces, and let $A \in \mathcal{N}(X,Y), B \in \mathcal{N}^{-1}(X,Y)$.*

If one of the following compactness conditions holds

(i) *Y is compact; or*
(ii) *X is compact; or*
(iii) *there exist a compact subset K of X and a compact convex subset C of Y with $A(x) \cap C \neq \emptyset$, for all $x \in X \setminus K$;*

then $A \cap B \neq \emptyset$.

Proof. Let it be made clear, first, that $(i) \Longrightarrow (iii)$ and $(ii) \Longrightarrow (iii)$. Indeed, if Y is compact, take $C = Y$ and $K = \emptyset$ in (iii). If X is compact, take $K = X$ and (iii) is vacuously satisfied. Moreover, due to Lemma 12, (iii) can be reduced to (i). Indeed, assume that (iii) holds. Lemma 12 implies the existence of a convex finite polytope $P \subset Y$ and a continuous mapping $s : K \longrightarrow Y$ with $s(x) \in A(x) \cap P$ for all $x \in K$. Now, if $x \in X \setminus K, A(x) \cap C \neq \emptyset$ where $C \subset Y$ is convex and compact. Consider the convex hull $\hat{Y} = conv(P \cup C)$, a compact subset of Y (the convex envelope of two compact convex sets in a topological vector space is also compact convex). It is clear that the map $\hat{A} : X \longrightarrow 2^{\hat{Y}}$ given by $\hat{A}(x) := A(x) \cap \hat{Y}, x \in X$, defines a von Neumann relation, i.e., $\hat{A} \in \mathcal{N}(X, \hat{Y})$. Also, the mapping $\hat{B} : X \longrightarrow 2^{\hat{Y}}$ given by $\hat{B}(x) := B(x) \cap \hat{Y}, x \in X$, verifies $\hat{B} \in \mathcal{N}^{-1}(X, \hat{Y})$. A coincidence for the pair $(\hat{A}, \hat{B})$ is also a coincidence for (A, B).

It suffices, thus, to show that $A \cap B \neq \emptyset$ under hypothesis (i), i.e., when Y is compact. Since for all $y \in Y, B^{-1}(y) \neq \emptyset$, it follows that $\{B(x)\}_{x \in X}$ forms an open cover of Y. Similarly, $\{A^{-1}(y)\}_{y \in Y}$ is an open cover of X. Let $\{B(x_i)\}_{i=1}^n$ be a finite subcover of Y, let $D := conv(\{x_1, \dots, x_n\})$, a convex compact subset of X, and let $\{A^{-1}(y_j)\}_{j=1}^m$ be an open subcover of D. Consider the convex compact subset $M := conv(\{y_1, \dots, y_m\})$ of Y. M can be covered by a subfamily of $\{B(x_i)\}_{i=1}^n$, which, for simplicity we also denote $\{B(x_i)\}_{i=1}^n$. We can assume with no loss of generality that $\{B(x_i)\}_{i=1}^n$ and $\{A^{-1}(y_j)\}_{j=1}^m$ are minimal covers of M and D, respectively. That is, for any $k \in \{1, \dots, n\}, M \nsubseteq \bigcup_{i=1, i\neq k}^n B(x_i)$, and for any $l \in \{1, \dots, m\}, D \nsubseteq \bigcup_{j=1, j\neq l}^m A^{-1}(y_j)$. Consider the compact convex sets $M_i = M \setminus B(x_i)$ for for $i = 1, \dots, n$, and $D_j = D \setminus A^{-1}(y_j)$ for $j = 1, \dots, m$. The fact that $M \subseteq \bigcup_{i=1}^n B(x_i)$ is equivalent to $\bigcap_{i=1}^n M_i = \emptyset$, and $D \subseteq \bigcup_{j=1}^m A^{-1}(y_j)$ is equivalent to $\bigcap_{j=1}^m D_j = \emptyset$. The minimality of the covers $\{B(x_i)\}_{i=1}^n$ and $\{A^{-1}(y_j)\}_{j=1}^m$ amounts to $M \cap \bigcap_{i=1, i\neq k}^n M_i \neq \emptyset$ for any $k \in \{1, \dots, n\}$, and $D \cap \bigcap_{j=1, j\neq l}^m D_j \neq \emptyset$ for any $l \in \{1, \dots, m\}$. All conditions of Klee's Theorem (see Remark 4) are thus satisfied for both families of compact convex sets $\{M, M_1, \ldots, M_n\}$ and $\{D, D_1, \ldots, D_m\}$. Hence, $M \nsubseteq \bigcup_{i=1}^{n} M_i$ and $D \nsubseteq \bigcup_{j=1}^{m} D_j$. Let $y_0 \in M$ but $y_0 \notin M_i$ for all $i = 1, \ldots, n$, and let $x_0 \in D$ but $x_0 \notin D_j$ for all $j = 1, \ldots, m$. Clearly, $y_0 \in B(x_i) \Longleftrightarrow x_i \in B^{-1}(y_0)$ for all $i = 1, \ldots, n$, and $x_0 \in A^{-1}(y_j) \Longleftrightarrow y_j \in A(x_0)$ for all for all $j = 1, \ldots, m$. The sections $B^{-1}(y_0)$ and $A(x_0)$ being convex sets, it follows that $x_0 \in D = conv(\{x_1, \ldots, x_n\}) \subset B^{-1}(y_0)$ and $y_0 \in M = conv(\{y_1, \ldots, y_m\}) \subset A(x_0)$. The proof is finished as $(x_0, y_0) \in A \cap B$. □

4. Analytic Formulations and Applications

This section illustrates how the geometric results in the preceding section, Theorems 7, 10, and 13, are key in deriving a number of landmark results in functional analysis. Intersection theorems as well as fixed point and coincidence theorems have analytical formulations as solvability theorems for systems of nonlinear inequalities (see [1, 2, 3, 4, 9]). These analytical formulations are often more practical when it comes to applications. We start with the analytical formulation of the convex KKM Principle (equivalently, the fixed point theorem for von Neumann relations) and we derive from it, in a simple and straightforward way, two fundamental results.

4.1. Alternatives for Systems of Nonlinear Inequalities and Applications.

Theorems 7 and 10 can be expressed in terms of an alternative for nonlinear systems of inequalities *à la Ky Fan*.

Recall first the basic concepts of semicontinuity and quasiconvexity for real functions.

Definition 14. *A real function $f : X \longrightarrow \mathbb{R}$ defined on a subset X of a t.v.s is:*

(i) *quasiconvex if for all $\lambda \in \mathbb{R}$, the level set $\{x \in X; f(x) \leq \lambda\}$ is a convex subset of X;*
(ii) *quasiconcave if $-f$ is quasiconvex;*
(iii) *lower semicontinuous (l.s.c.) if for all $\lambda \in \mathbb{R}$, the level set $\{x \in X; f(x) \leq \lambda\}$ is a closed subset of X;*
(iv) *upper semicontinuous (u.s.c.) if $-f$ is l.s.c.*

Naturally, every convex functional is quasiconvex and the converse is false. Also, a real function on a topological space is continuous if and only if it is both upper and lower semicontinuous.

Theorem 15. *Let X be a convex subset of a t.v.s. E, Y a non-empty subset of X, and $f : X \times Y \longrightarrow \mathbb{R}$ a function satisfying*

(i) *$x \mapsto f(x, y)$ is l.s.c. and quasiconvex on X, for each fixed $y \in Y$.*
(ii) *$y \mapsto f(x, y)$ quasiconcave on Y, for each fixed $x \in X$.*

Assume that for a given $\lambda \in \mathbb{R}$, there exist a compact subset K of X and a convex compact subset D of Y such that for all $x \in X \setminus K$, there exists $y \in D$ with $f(x, y) > \lambda$.

Then the following alternative holds:

(A) *there exists $x_0 \in X$ such that $f(x_0, x_0) > \lambda$, or*
(B) *there exists $\bar{x} \in Y$ such that $f(\bar{x}, y) \le \lambda$, for all $y \in Y$.*

Consequently, when $\lambda = \sup_{x\in X} f(x, x)$, (A) is impossible and

$$\inf_{x\in X} \sup_{y\in Y} f(x, y) \le \sup_{x\in X} f(x, x).$$

Proof. Let $A(x) := \{y \in Y : f(x, y) > \lambda\}, x \in X$. All hypotheses of Theorem 10 are satisfied except, possibly, the non-emptiness of the sections $A(x)$. Thus, either $A(x) \neq \emptyset$, for all $x \in X$, hence A is a von Neumann relation, and therefore has a fixed point ((A) holds), or $A(\bar{x}) = \emptyset$ for some $\bar{x} \in X$, i.e., A is not a von Neumann relation and (B) is satisfied. □

This is a particular instance of the celebrated *Infsup Inequality of Ky Fan* with a weaker compactness condition.

Landmark theorems of nonlinear functional analysis follow immediately from Theorem 15; therefore, indirectly, from the separation of closed convex sets in finite dimension (Proposition 2). We refer to H. Brézis [6] for an account and applications of the next two fundamental results.

Corollary 16. *(Mazur-Schauder Theorem). Let X be a non-empty closed convex subset of a reflexive Banach space E and let $\varphi : X \longrightarrow \mathbb{R}$ be a lower semicontinuous, quasiconvex and coercive (i.e. $\lim_{||x||\to\infty} \varphi(x) = \infty$) functional. Then φ achieves its minimum on X.*

Proof. Let $\lambda = 0$, $Y = X$, and $f(x, y) = \varphi(x) - \varphi(y)$ in Theorem 15. Let K be the intersection of X with a closed ball with radius $M > 0$ centered at the origin of E and such that if $x \in X$ with $\|x\| > M$ then $\varphi(x) > \varphi(y)$ for some $y \in K$. Such a non-empty set K exists due to the coercivity of φ. Since E is reflexive, K is weakly compact. One readily verifies that the hypotheses of Theorem 15 with $X, Y, f, K, D = K$, and $\lambda = 0$ all hold: f is l.s.c. and quasiconvex in x, and quasiconcave in y. Clearly, possibility (A) of Theorem 15 cannot hold. Hence (B) is true: there exists $\bar{x} \in X$ such that $f(\bar{x}, y) = \varphi(\bar{x}) - \varphi(y) \le 0$, for all $y \in X$.

We now derive from the nonlinear alternative in Theorem 15 the celebrated theorem of Stampacchia for variational inequalities. Recall that given a normed space E, a form $a : E \times E \longrightarrow \mathbb{R}$ is said to be

(i) *bilinear* if it is linear in each of its arguments;
(ii) *continuous* if there exists a constant $C > 0$ with $|a(x, y)| \le C||x||||y||$ for all $x, y \in E$; and

(iii) *coercive* if there exists a constant $\alpha > 0$ with $a(x,x) \geq \alpha ||x||^2$ for all $x \in E$.

$\square$

Corollary 17. *(Stampacchia Theorem). Let E be a reflexive Banach space, $a : E \times E \longrightarrow \mathbb{R}$ be a continuous and coercive bilinear form, and let $\ell : E \longrightarrow \mathbb{R}$ be a bounded linear functional. Given a non-empty closed and convex subset X in E, there exists a unique $\bar{x} \in X$ such that $a(\bar{x}, \bar{x} - y) \leq \ell(\bar{x}) - \ell(y)$ for all $y \in X$.*

Proof. For the existence, we apply Theorem 15 to $f : X \times X \longrightarrow \mathbb{R}$ defined by $f(x,y) := a(x, x-y) - \ell(x-y), (x,y) \in X \times X, \lambda = 0$, $D = \{y_0\}$ with $0 \neq y_0 \in X$ arbitrary, and $K := \{x \in X : \|x\| \leq M\}$ where

$$M := \frac{1}{2}(\beta + \sqrt{\beta^2 + 4\gamma}),$$

$\beta = (C||y_0|| + ||\ell||)/\alpha$ and $\gamma = ||\ell|| ||y_0||/\alpha$.

Indeed, first note that if E is equipped with the weak topology, then $f(x,y)$ is l.s.c. and quasiconvex in x and quasiconcave in y (it is in fact linear and continuous for the norm topology in both arguments). Since X is closed and convex, it follows that K is a closed, convex and bounded, hence weakly compact, subset of X. D is obviously a weakly compact subset of X. Note now that if $f(x, y_0) \leq 0$ for any given $x \in X$, i.e., $a(x,x) \leq a(x, y_0) + \ell(x - y_0)$, then $\|x\|$ satisfies a quadratic inequality and is bounded above by M:

$$\begin{array}{rl} & \alpha||x||^2 \leq a(x,x) \leq C\|x\|\|y_0\| + \|\ell\|\|x\| + \|\ell\|\|y_0\| \\ \Longrightarrow & \alpha||x||^2 - (C\|y_0\| + \|\ell\|)\|x\| - \|\ell\|\|y_0\| \leq 0 \\ \Longleftrightarrow & ||x||^2 - \beta\|x\| - \gamma \leq 0 \\ \Longrightarrow & \|x\| \leq \frac{1}{2}(\beta + \sqrt{\beta^2 + 4\gamma}) = M. \end{array}$$

Consequently, if $x \in X, \|x\| > M$, then $f(x, y_0) > 0$ and the compactness condition in Theorem 15 is satisfied. Since $f(x,x) = 0$ for any $x \in X$, (A) of Theorem 15 is impossible, and (B) holds, i.e., $f(\bar{x}, y) = a(\bar{x}, \bar{x} - y) - \ell(\bar{x}) + \ell(y) \leq 0$ for some $\bar{x} \in X$ and all $y \in X$ and the proof of the existence is complete.

The uniqueness follows at once from the bilinearity and the coercivity of the form a as follows: if $a(\bar{x}_i, \bar{x}_i - y) - \ell(\bar{x}_i) + \ell(y) \leq 0$ for two elements $\bar{x}_i \in X, i = 1, 2$, and all $y \in X$, then adding $a(\bar{x}_1, \bar{x}_1 - \bar{x}_2) \leq \ell(\bar{x}_1) - \ell(\bar{x}_2)$ to $a(\bar{x}_2, \bar{x}_2 - \bar{x}_1) \leq \ell(\bar{x}_2) - \ell(\bar{x}_1)$ gives $0 \leq \alpha\|\bar{x}_1 - \bar{x}_2\|^2 \leq a(\bar{x}_1 - \bar{x}_2, \bar{x}_1 - \bar{x}_2) \leq 0$, i.e., $\bar{x}_1 = \bar{x}_2$. $\square$

The coincidence $(\mathcal{N}, \mathcal{N}^{-1})$ (Theorem 13) can be expressed in analytical terms as a second alternative for nonlinear systems of inequalities as follows.

Theorem 18. *Let X and Y be two convex subsets of topological vector spaces and let $f, g : X \times Y \longrightarrow \mathbb{R}$ be two functions satisfying*

(i) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in X \times Y$*;*
(ii) $x \mapsto f(x, y)$ *is quasiconcave on X, for each fixed $y \in Y$;*
(iii) $y \mapsto f(x, y)$ *is l.s.c. and quasiconvex on Y, for each fixed $x \in X$;*
(iv) $x \mapsto g(x, y)$ *is u.s.c. and quasiconcave on X, for each fixed $x \in X$;*
(v) $y \mapsto g(x, y)$ *is quasiconvex on Y, for each fixed $x \in X$.*
(vi) *Given $\lambda \in \mathbb{R}$ arbitrary, assume that either Y is compact, or X is compact, or there exist a compact subset K of X and a convex compact subset C of Y such that for any $x \in X \setminus K$ there exists $y \in C$ with $g(x, y) < \lambda$.*

Then one of the following statements holds:

A) *there exists $\bar{x} \in X$ such that $g(\bar{x}, y) \geq \lambda$, for all $y \in Y$; or*
B) *there exists $\bar{y} \in Y$ such that $f(x, \bar{y}) \leq \lambda$, for all $x \in X$.*

Proof. Simply apply Theorem 13 to $A, B \subset X \times Y$ defined as

$$A := \{(x, y) : g(x, y) < \lambda\} \text{ and } B := \{(x, y) : f(x, y) > \lambda\}.$$

Note that in view of (i) a coincidence between A and B is impossible as it yields $\lambda < \lambda$. Since all hypotheses of Theorem 13 are satisfied save for $A(x) \neq \emptyset$ for all $x \in X$ and $B^{-1}(y) \neq \emptyset$ for all $y \in Y$, it follows that either $A(\bar{x}) = \emptyset$ for some $\bar{x} \in X$ (thesis (A)) or $B^{-1}(\bar{y}) = \emptyset$ for some $\bar{y} \in Y$ (thesis (B)). $\square$

Remark 19. *Theorem 18 implies $\alpha = \sup_X \inf_Y g(x, y) \geq \inf_Y \sup_X f(x, y) = \beta$.*

Indeed, assuming that $\alpha < \beta < \infty$, let λ be an arbitrary but fixed real number strictly between α and β. By Theorem 18, either there exists $\bar{y} \in Y$ such that $f(x, \bar{y}) \leq \lambda$, for all $x \in X$ thus $\beta \leq \lambda < \beta$ which is impossible, or there exists $\bar{x} \in X$ such that $g(\bar{x}, y) \geq \lambda$, for all $y \in Y$ thus $\alpha \geq \lambda > \alpha$ which is absurd. Hence, $\alpha \geq \beta$.

Maurice Sion's formulation of the von Neumann Minimax Theorem follows immediately with $f = g$ (see [1]).

Corollary 20. *(Sion-von Neumann Minimax Theorem). Let X and Y be convex subsets of topological vector spaces and let f be a real function on $X \times Y$ such that:*

(i) $x \mapsto f(x, y)$ *is quasiconcave and u.s.c. on X, for each fixed $y \in Y$;*
(ii) $y \mapsto f(x, y)$ *is quasiconvex and l.s.c. on Y, for each fixed $x \in X$.*

Assume that either X is compact or Y is compact. Then:

$$\alpha = \sup_X \inf_Y f(x, y) = \inf_Y \sup_X f(x, y) = \beta$$

Proof. The inequality $\alpha \leq \beta$ is always true and $\alpha \geq \beta$ follows from Remark 19. $\square$

Remark 21. *If both X and Y are compact, the infsup equality in Corollary 20 is a minmax equality and is equivalent to the existence of a saddle point (x_0, y_0) for the function $f(x, y)$, i.e., $f(x, y_0) \leq f(x_0, y)$, for all $(x, y) \in X \times Y$.*

We end this section with a short proof of the Markov-Kakutani Fixed Point Theorem for abelian families of continuous affine mappings in linear topological spaces having separating duals[1].

Recall that a mapping ϕ from a convex set X into a vector space is said to be *affine* if and only if $\phi(\sum_{i=1}^n \lambda_i x_i) = \sum_{i=1}^n \lambda_i \phi(x_i)$ for any convex combination $\sum_{i=1}^n \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1$, in X. The key ingredient is the following fixed point property for continuous affine transformations of a compact convex set.

Corollary 22. *Let X be a non-empty compact convex subset of a t.v.s. E with separating dual E' and let $\phi : X \longrightarrow X$ be a continuous affine mapping. Then ϕ has a fixed point.*

Proof. The proof is a simplification of the treatment in [9]. Define $f : X \times E' \longrightarrow \mathbb{R}$ by $f(x, \ell) = \ell(\phi(x) - x), (x, \ell) \in X \times E'$. It suffices to prove the existence of $x_0 \in X$ such that $f(x_0, \ell) \leq 0$, for all $\ell \in E'$, for this would imply $\ell(\phi(x_0) - x_0) = 0$, for all $\ell \in E'$, i.e., $\phi(x_0) - x_0 = 0$ and the proof is complete.

This amounts to showing that $\bigcap_{\ell \in E'} A(\ell) \neq \emptyset$ for the relation $A := \{(\ell, x) \in E' \times X : f(x, \ell) \leq 0\}$.

Since for each fixed $\ell \in E'$, the function $f(x, \ell)$ is l.s.c. in x, then each $A(\ell)$ is a closed, hence compact, subset of X. It suffices, therefore, to show that the collection $\{A(\ell) : \ell \in E'\}$ has the finite intersection property. Consider, to this aim, a finite collection of bounded linear functionals $L := \{\ell_1, \ldots, \ell_n\} \subset E'$, and let $Y = conv(L)$, a convex compact subset of E'. The restriction of $f(x, \ell)$ to $X \times Y$ is obviously u.s.c. and quasiconcave in x and l.s.c. and quasiconvex in ℓ. Since both X and Y are compact and convex, it follows from Remark 21 that there exists $(x_0, \ell_0) \in X \times Y$ with $f(x_0, \ell) \leq f(x, \ell_0)$ for all $(x, \ell) \in X \times Y$, i.e., $\ell(\phi(x_0) - x_0) \leq \ell_0(\phi(x) - x)$ for all $(x, \ell) \in X \times Y$. Let $\hat{x}$ be such that $\ell_0(\hat{x}) = \max_{x \in X} \ell_0(x)$. Since $\phi(\hat{x}) \in X$, it follows that $\ell_0(\phi(\hat{x}) - \hat{x}) \leq 0$ and, consequently, $\ell(\phi(x_0) - x_0) \leq 0$,

[1]A t.v.s. *E has separating dual* if for each $x \in E, x \neq 0$, there exists a bounded linear form $\ell \in E'$, the topological dual of E, such that $\ell(x) \neq 0$. Locally convex topological vector spaces have separating duals. Sequence spaces $\ell^p, 0 < p < 1$, and Hardy spaces $H^p, 0 < p < 1$, are instances of non-locally convex spaces with separating duals.

for all $\ell \in Y$, in particular, $\ell_i(\phi(x_0) - x_0) = f(x_0, \ell_i) \leq 0$ for all $\ell_i \in L$, and the proof is complete. □

The Markov-Kakutani follows by a standard compactness argument. Recall that a family $\mathcal{F}$ of mappings is said to be *abelian* if $\phi_1\phi_2 = \phi_2\phi_1$ for all $\phi_1, \phi_2 \in \mathcal{F}$.

Corollary 23. *(Theorem of Markov-Kakutani). Let X be a non-empty compact convex subset of a t.v.s. E with separating dual E' and let $\mathcal{F}$ be an abelian family of continuous affine transformations from X into itself. Then, there exists $x_0 \in X$ such that $\phi(x_0) = x_0$ for every $\phi \in \mathcal{F}$.*

Proof. For any given $\phi \in \mathcal{F}$, let $Fix(\phi)$ be the set of its fixed points. We show that $\bigcap_{\phi\in\mathcal{F}} Fix(\phi) \neq \emptyset$. Clearly, for each $\phi \in$ Fourier$Fix(\phi)$ is non-empty (by Corollary 21), convex (as ϕ is affine), and closed hence compact in X. It suffices to show that the family $\{Fix(\phi) : \phi \in \mathcal{F}\}$ has the finite intersection property, i.e., $\bigcap_{i=1}^{n} Fix(\phi_i) \neq \emptyset$ for any $\{\phi_1, \ldots, \phi_n\} \subset \mathcal{F}$. The proof is by induction on n. For $n = 1$, clearly $Fix(\phi_1) \neq \emptyset$ (Corollary 21). Assume that the statement is true for any family $\{\phi_1, \ldots, \phi_k\} \subset \mathcal{F}$ with $k = n - 1$ and let $\{\phi_1, \ldots, \phi_n\} \subset \mathcal{F}$ be arbitrary. For any $x \in \bigcap_{i=1}^{n-1} Fix(\phi_i)$, $\phi_n(x) = \phi_n(\phi_i(x)) = \phi_i(\phi_n(x))$ for all $i = 1, \ldots, n-1$, i.e., $\phi_n(x) \in \bigcap_{i=1}^{n-1} Fix(\phi_i)$. Thus ϕ_n maps the non-empty compact convex set $\bigcap_{i=1}^{n-1} Fix(\phi_i)$ into itself. By Corollary 21 again, it has a fixed point $\bar{x} = \phi_n(\bar{x}) \cap \bigcap_{i=1}^{n-1} Fix(\phi_i)$, i.e., $\bar{x} \in \bigcap_{i=1}^{n} Fix(\phi_i)$. □

5. Concluding Remarks

It is well established that the Markov-Kakutani Fixed Point Theorem implies the Hahn-Banach Theorem (Kakutani [16]). The two results are indeed equivalent (for a short and elegant proof of the converse, see D. Werner [21]). Since we derived here the convex KKM Theorem from the theorem on the separation of convex sets, we have thus established the equivalence of the Hahn-Banach Theorem, Klee's Intersection Theorem, the convex KKM Theorem, the fixed point theorem for von Neumann relations, the Sion-von Neumann Minimax Theorem, and the Markov-Kakutani Fixed Point Theorem.

Although the convex KKM Theorem is a particular instance of the KKM Principle of Ky Fan, and since the fixed point and coincidence properties in Theorems 9 and 12 are special cases of similar results for so-called F and F^* maps (see [2, 3, 4]), the interest here resides in the use of simple arguments of convexity rather than the Brouwer Fixed Point Theorem or Sperner's Lemma. It would be most interesting to know if the Ky Fan KKM Principle (or any of its equivalent results) can be derived directly from the convex KKM Theorem. In other words, can any of the question marks below be settled? (The smaller arrows are established; FPT stands for Fixed Point Theorem.)

$$\left[\begin{array}{ccccc}
\text{Convex Klee} & & \text{Convex KKM} & & \text{FPT for } \mathcal{N} \\
\text{(Proposition 3)} & \rightleftarrows & \text{(Theorem 7)} & \rightleftarrows & \text{(Theorem 10)} \\
\uparrow\downarrow ? & \nwarrow\searrow ? & \uparrow\downarrow ? & ?\swarrow\nearrow & \uparrow\downarrow ? \\
\text{KKM Lemma} & \rightleftarrows & \text{Ky Fan KKM} & \rightleftarrows & \text{Browder-Ky Fan FPT}
\end{array}\right]$$

Let us point out, for the reader's benefit that the following equivalences have been established in [15]:

$$\left[\begin{array}{ccccc}
\text{Brouwer FPT} & & \rightleftarrows & & \text{Topological Klee} \\
 & \nwarrow\searrow & & \swarrow\nearrow & \\
 & & \text{Ky Fan KKM} & &
\end{array}\right]$$

Here, the "topological Klee Theorem" [15] reads:

Theorem 24. *A family of n closed convex sets in a topological vector space has a non-empty intersection if and only if the union of the n sets is $(n-2)$-connected and the intersection of every $n-1$ of them is non-empty.*

This topological version of Klee's theorem yields the equivalent formulation of the Brouwer Fixed Point Theorem:

the n-sphere S^n is not n-connected.

Indeed, the $n-$dimensional faces of the $(n+1)-$simplex Δ^{n+1} form a family of $n+2$ closed convex sets in $\mathbb{R}^{n+2}$. Moreover, every intersection of $n+1$ of closed convex sets is non-empty, but the whole intersection is empty. Hence, their union - which consists of the boundary $\partial\Delta^{n+1}$ - is not $n-$connected. Since $\partial\Delta^{n+1}$ is homeomorphic to S^n, it follows that S^n is not n-connected. This establishes the implication: topological Klee Theorem $\Longrightarrow$ Brouwer FPT. The equivalence Brouwer FPT $\Longleftrightarrow$ Ky Fan KKM Principle is well-established (see e.g., [9]).

Also, since every convex set in a topological vector space is contractible, hence $n-$connected for every $n \geq 0$, the topological Klee Theorem implies Proposition 3. Does the converse hold true?

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