ON THE PRIME WHALES OF A BOOLEAN ALGEBRA

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Abstract. In this note, we introduce objects called prime whales and use them to represent a Boolean algebra as an algebra of sets in a way that is analogous to Stone's Representation Theorem. We also characterize the existence of prime whales in terms of the existence of prime ideals.

1. Introduction

It has been a little more than seventy years since Marshall Stone's work on Boolean algebras was published in [6]. His results have been used in far reaching applications in various areas of mathematics including measure theory, topology, and logic.

One version of Stone's Representation Theorem affirms that any Boolean algebra B can be identified with an algebra of sets. Standard proofs of Stone's Theorem utilize the powerset of the prime ideals of B as in Theorem 10.22 in [4] or the powerset of the ultrafilters of B as in Theorem 2.1 in [5]. The topological version of Stone's Theorem characterizes the representative algebra of sets as a certain zero dimensional topological space known as a Stone space (see chapter 9 section 9 in [1]). Whether the representation involves prime ideals or ultrafilters, the Boolean Prime Ideal Theorem must be invoked. This assures that one has enough prime ideals (or ultrafilters) for a faithful representation of the Boolean algebra.

The authors in [2] and [3] utilize objects called *whales* inside specific Boolean algebras to construct the universal completion and the orthocompletion for certain vector lattices and lattice-ordered groups.

In this paper, we give an alternative to Stone's representation utilizing what we call prime whales. Just as the Boolean Prime Ideal Theorem is needed in a representation using prime ideals, we will need assurance that prime whales indeed exist.

2. Background

For unexplained terminology and notation, we refer the reader to the excellent introductory text [4]. Let B denote a Boolean algebra and let $x \in B$. The symbol $\bar{\uparrow} x$ will denote all elements in the principal order filter (or up-set) generated by x *except* the largest element which we denote by $\mathbf{1}$. *Whales* are defined to be order ideals that have supremum equal to $\mathbf{1}$. A subset $\mathcal{A}$ of a Boolean algebra B is a *prime whale* if $\mathcal{A}$ is a whale such that for every $x \in B$, exactly one of the sets $\overline{\uparrow}x$ or $\overline{\uparrow}x'$ is contained in $\mathcal{A}$. Since $\overline{\uparrow}\mathbf{1} = \varnothing$, we have that $\overline{\uparrow}\mathbf{1}$ is a member of every prime whale and $\overline{\uparrow}\mathbf{1}' = \overline{\uparrow}\mathbf{0}$ is never a member of a prime whale.

In the rest of this paper, we assume that any Boolean algebra has at least 8 elements. If a Boolean algebra has fewer than 8 elements, then whales exist but prime whales do not. For example, in a Boolean algebra with 4 elements, any order ideal with supremum equal to $\mathbf{1}$ must contain both $\overline{\uparrow}x$ and $\overline{\uparrow}x'$ when x is an atom. Theorem 2.5 in this paper provides a characterization for the existence of prime whales in all Boolean algebras with 8 elements or greater.

We will also assume that given a Boolean algebra B and $b \in B^+$, there exists a prime whale that contains $\overline{\uparrow}b$. This is analogous to finding a prime ideal that contains a given proper ideal. We acknowledge that prime whales are more complicated than prime ideals and thus would not be the perfect candidate for representing a Boolean algebra. Theorem 2.5 shows however that prime whales and prime ideals must mutually exist and is therefore another characterization of prime ideals.

The following lemma provides insight into the structure of a prime whale and will be utilized later.

Lemma 2.1. *Let $\mathcal{A}$ be a prime whale in the Boolean algebra B. Let $\mathcal{I}$ denote the ideal generated by the set $B \setminus (\mathcal{A} \cup \mathbf{1})$. Then $\mathcal{I}$ is a proper ideal.*

Proof. Let $x, y \in B \setminus (\mathcal{A} \cup \mathbf{1})$. Then $x \vee y < \mathbf{1}$, otherwise $\mathcal{A}$ avoids both $\overline{\uparrow}x$ and $\overline{\uparrow}x'$. Reasoning inductively, we see that $\vee A < \mathbf{1}$ for any finite set $A \subseteq \mathcal{A}$. It follows that $\mathcal{I}$ is a proper ideal. □

Before proceeding, we give a concrete example of a prime whale.

Example 2.2. *Let B be an atomic Boolean algebra and let x be the complement of an atom. Note that $(\uparrow x) = \{x, \mathbf{1}\}$. To see that $B \setminus (\uparrow x)$ is a prime whale, consider any other element $y \in B$. By Lemma 2.4b in [5], either $x' \leq y$ or $x' \leq y'$ but not both. Without loss of generality, assume that $x' \leq y$. Then $\overline{\uparrow}y \subseteq \overline{\uparrow}x' \subseteq B \setminus (\uparrow x)$. We also have that $x \geq y'$ and thus $B \setminus (\uparrow x)$ cannot contain $\overline{\uparrow}y'$.*

Conversely, one can prove that a prime whale in an atomic Boolean algebra must be of the form $B \setminus (\uparrow x)$ where x' is an atom.

Given a Boolean algebra B and an element $b \in B$, we denote the set of all prime whales that contain $\overline{\uparrow}b$ by X_b. We desire a correspondence between the Boolean algebra B and the powerset all prime whales denoted

$\mathcal{P}(\mathbb{X})$. The obvious candidate for such a mapping is $b \to X_b$. The first step toward the desired correspondence is the following lemma.

Lemma 2.3. *Let B be a Boolean algebra. If $a, b \in B$, then $X_{a\wedge b} = X_a \cap X_b$.*

Proof. According to the definition, a prime whale cannot contain $\overline{\uparrow}\mathbf{0}$, but must always contain $\overline{\uparrow}\mathbf{1}$. Thus, we have that $X_{\mathbf{0}} = \varnothing$ and $X_{\mathbf{1}} = \mathcal{P}(\mathbb{X})$. Suppose that $a \wedge b = \mathbf{0}$. It follows that $a' \geq b$ and thus, any prime whale that contains $\overline{\uparrow}b$ would necessarily contain $\overline{\uparrow}a'$, while at the same time avoiding $\overline{\uparrow}a$. Therefore, $X_0 = X_{a\wedge b} = X_a \cap X_b = \varnothing$. If $a \wedge b > 0$, then let $\mathcal{A} \in X_a \cap X_b$. Let $y \in B \setminus (\mathcal{A} \cup \mathbf{1})$. Note that $\mathcal{A}$ avoids $\overline{\uparrow}y$ and therefore contains $\overline{\uparrow}y'$. It follows that $y \vee a = y \vee b = \mathbf{1}$ for if not, y would be an element of $\mathcal{A}$. From the latter fact we obtain $y \geq a'$ and $y \geq b'$ which implies $y' \leq a$ and $y' \leq b$. Thus, $y' \leq a \wedge b$.
Since $\mathcal{A}$ must contain $\overline{\uparrow}y'$, it follows that $\overline{\uparrow}(a \wedge b) \in \mathcal{A}$ which implies $X_{a\wedge b} \supseteq X_a \cap X_b$. It is evident that $X_{a\wedge b} \subseteq X_a \cap X_b$. □

Theorem 2.4. *The mapping $b \to X_b$ is a Boolean algebra embedding of B into $\mathcal{P}(\mathbb{X})$.*

Proof. By the definition of a prime whale, $X_{b'}$ is the complement of X_b in the power set algebra $\mathcal{P}(\mathbb{X})$. By Lemma 2.3 above and Lemma 4.17 in [4], the mapping $b \to X_b$ is a Boolean homomorphism.

Our last task is to show that the mapping $b \to X_b$ is injective. Suppose that $a, b \in B^+$ and $a \neq b$. If $a \wedge b = \mathbf{0}$, then any prime whale containing $\overline{\uparrow}b$ must avoid $\overline{\uparrow}a$. Now if $a \wedge b > \mathbf{0}$, then either $a' \wedge b \in B^+$ or $a \wedge b' \in B^+$. Without loss of generality assume that $a' \wedge b > \mathbf{0}$. Then there exists a prime whale that contains $\overline{\uparrow}(a' \wedge b)$ and hence both $\overline{\uparrow}a'$ and $\overline{\uparrow}b$. It follows that $X_a \neq X_b$. □

Theorem 2.4 is an alternative proof of Stone's Representation Theorem as long as the Boolean algebra in question has sufficiently many prime whales. Next, in Theorem 2.5, we see that prime whales exist if and only if prime ideals exist.

Theorem 2.5. *Let B be a Boolean algebra. Then the following are equivalent.*

- *i.* *Given a proper ideal $J \subseteq B$, there exists a prime ideal $I \in B$ such that $J \subseteq I$.*
- *ii.* *Given $b \in B^+$, there exists a prime whale $\mathcal{A}$ such that $\overline{\uparrow}b \subseteq \mathcal{A}$.*

Proof. To show that i implies ii, let $b \in B^+ \setminus \{\mathbf{1}\}$ and let I be a prime ideal containing the principal ideal generated by b'. Now if b' is dual to an atom, then as in Example 2.2, $\mathcal{A} = B \setminus \{b', \mathbf{1}\}$ is a prime whale containing $\overline{\uparrow}b$. If b' is not dual to an atom, then set

$$\mathcal{W} = \{x \in B : x \leq b'\} \cup \{x \in B : x \leq z \text{ where } z \in \overline{\uparrow}b\}.$$

Note that $\mathcal{W}$ is a whale that does not contain all of $\overline{\uparrow}b'$ and therefore does not contain all of I.

Now set

$$\mathcal{A} = \bigcup\{\mathcal{W} \;:\; \mathcal{W} \text{ is a whale that does not contain all of } I\}.$$

By Lemma 1 in [2], $\mathcal{A}$ is a whale. Also note that $\overline{\uparrow}b \subseteq \mathcal{A}$. To see that $\mathcal{A}$ is a prime whale, let $g \in B^+$. Without loss of generality, assume $g' \in I$. It remains to show that $\overline{\uparrow}g \subseteq \mathcal{A}$. Once again, if g' is dual to an atom, then $B \setminus \{g', \mathbf{1}\}$ is a prime whale containing $\overline{\uparrow}g$. If g' is not dual to an atom, then as before

$$\mathcal{W} = \{x \in B : x \leq g'\} \cup \{x \in B : x \leq z \text{ where } z \in \overline{\uparrow}g\}$$

is a whale containing $\overline{\uparrow}g$. It follows that $\overline{\uparrow}g \subseteq \mathcal{A}$.

To show that *ii* implies *i*, let $\mathcal{A}$ be a prime whale. Set I equal to the ideal generated by the set $B \setminus (\mathcal{A} \cup \mathbf{1})$. We claim that I is a prime ideal. If I contains neither x nor x' for some $x \in B$, then both $\overline{\uparrow}x$ and $\overline{\uparrow}x'$ are contained in $\mathcal{A}$ which contradicts the fact that $\mathcal{A}$ is a prime whale. By Lemma 2.1, the ideal I must be proper and therefore cannot contain both x and x'. By Theorem 10.12 in [4], I is a prime ideal. □

Remark 2.6. *It is important to observe that prime whales are indeed different than prime ideals. For example in a finite Boolean algebra of size* 2^n*, prime ideals have size* 2^{n-1}*. This is easy to see since every prime ideal is complementary to an ultrafilter of the same size. If* $n > 2$*, then the prime whales of a Boolean algebra of size* 2^n *have* $2^n - 2$ *elements. For example,* $\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$ *is a prime whale in the powerset algebra of* $\{1,2,3\}$.

The axiom of choice implies the Boolean prime ideal theorem but not conversely (see Proposition 2.16 in [5]). The reader is invited to examine chapter 10 of [4] for a discussion of axioms equivalent to and implied by the axiom of choice and the Boolean Prime Ideal Theorem.

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