A CONTINUOUS LINEAR BIJECTION FROM ℓ^2 ONTO A DENSE SUBSET OF ℓ^2 WHOSE INVERSE IS EVERYWHERE UNBOUNDEDLY DISCONTINUOUS

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Abstract. In a prior paper [1] we showed that there is a nonlinear, continuous, dense bijection from ℓ^2 onto a subset of ℓ^2 whose inverse is everywhere unboundedly discontinuous. We now show that there is a linear, continuous, dense bijection whose inverse is everywhere unboundedly discontinuous.

This note is motivated by term-by-term integration of Fourier series. F, as defined below, is essentially the term-by-term integral of a Fourier series. Roughly, the work below may be used to show that a term-by-term Fourier integral is the limit of the term-by-term Fourier integrals of a divergent set of functionals. Indeed, the same geometric results which were displayed by the function studied in [1] also apply to term-by-term Fourier integration. The details are available directly from the author. The definition of *everywhere unboundedly discontinuous*, an example of a real valued everywhere unboundedly discontinuous function, and geometric consequences of everywhere unboundedly discontinuous inverses of continuous, dense bijections may be found in [1], of which this note is a short extension.

For

$$a = \{a_n\}_{n=1}^{\infty} \in \ell^2, \text{ let } F(a) \overset{def}{=} \left(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \quad \ldots \quad , \frac{a_n}{n}, \quad \ldots \quad \right). \tag{1}$$

If

$$x \in F(\ell^2), F^{-1}(x) = (x_1, 2x_2, \quad \ldots \quad , nx_n, \quad \ldots \quad). \tag{2}$$

As the set of finite sequences is dense in ℓ^2 and any finite sequence is in the range of F, $F(\ell^2)$ is dense in ℓ^2. By inspection, each of F and F^{-1} is linear. The linear contraction F is continuous at 0 and hence everywhere. As F^{-1} is linear, F^{-1} is everywhere discontinuous if and only if F^{-1} is discontinuous at 0. Likewise, by a similar argument based upon linearity and translations, a linear operator is everywhere unboundedly discontinuous

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if and only if the linear operator is unboundedly discontinuous anywhere. Thus,

Lemma 1. *F^{-1} is everywhere unboundedly discontinuous.*

Proof. $F(0) = F^{-1}(0) = 0$. We define q_n by $q_1 = (1, 0, 0, \ldots), q_2 = \left(0, \frac{1}{\sqrt{2}}, 0, \ldots\right)$, $q_3 = \left(0, 0, \frac{1}{\sqrt{3}}, 0, \ldots\right)$, etc.

$$\|q_n - F(0)\| = \frac{1}{\sqrt{n}} \text{ and } \left\|F^{-1}(q_n) - F^{-1}(0)\right\| = \sqrt{n}.$$

Thus, as $n \to \infty$,

$$\|q_n\| \to 0, \text{ but } \left\|F^{-1}(q_n)\right\| \to \infty.$$

That is, F^{-1} is unboundedly discontinuous at 0, which, as F^{-1} is linear, assures that F^{-1} is everywhere unboundedly discontinuous. □

References

[1] S. H. Creswell, *A continuous bijection from ℓ^2 onto a subset of ℓ^2 whose inverse is everywhere unboundedly discontinuous, with an application to packing of balls in ℓ^2* Missouri J. Math. Sci., **23.1** (2011), 12–18.

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