A NOTE ON THE FUNDAMENTAL SOLUTION OF THE HEAT OPERATOR ON FORMS

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Abstract. An approximate solution of the heat equation on p-forms on an n-dimensional manifold is constructed. This is used to create a fundamental solution of the heat operator. It is shown there is a link between this solution and the generalized Gauss-Bonnet Theorem on manifolds.

1. Introduction

Let M be an n-dimensional compact oriented Riemannian manifold of class C^∞ without boundary. Denote by $\Lambda(M)$ the space of smooth exterior p-forms and $d\colon \Lambda^p \to \Lambda^{p+1}$ the exterior differentiation operator and $\delta\colon \Lambda^{p+1} \to \Lambda^p$ the adjoint of d with respect to the metric of M. The Laplace operator $\Delta = -(d\delta + \delta d)$ acts on p-forms for $0 \leq p \leq n$. The operator $\Delta\colon \Lambda^p \to \Lambda^p$ has an infinite sequence of eigenvalues $0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \to -\infty$. Each is repeated as many times as its multiplicity indicates. The corresponding sequence $\{\omega_n\}$ of eigenforms gives a complete orthonormal set in Λ^p with Riemannian inner product.

One of the objectives here is to construct a paramatrix for the heat equation for a p-form ω,

$$\left(\frac{\partial}{\partial t} - \Delta \right) \omega = 0. \tag{1.1}$$

This problem was discussed for functions by Minakshisundaram and Pleijel [4] and for forms it was studied by McKean and Singer [3] and also Patodi [5]. Let V be an open subset of M and $\omega(x,y)$ a $C^\infty(p,p)$ form on $V \times V$. For all $x \in M$, the Riemannian metric induces a natural isomorphism of $\Lambda^p T_x^*(M)$ onto the dual of this space. Thus there is a natural identification of $\Lambda^p T_x^*(M) \otimes \Lambda^p T_x^*(M)$ with $\mathrm{Hom}(\Lambda^p T^*(M), \Lambda^p T^*(M))$ and so for $x \in V$ and $v \in \Lambda^p T_x^*(M)$, $\omega(x,y)(v)$ is a smooth p-form on U [1,2].

For p-forms, an approximate solution which will be called $H_N^p(t,x,y)$ is constructed in a sufficiently small neighborhood of the diagonal in $V \times V$,

$t > 0$ by beginning with

$$H_N^p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \left(\sum_{i=0}^{N} t^i u^{i,p}(x,y) \right). \tag{1.2}$$

In (1.1), the variable r is the geodesic distance between x and y and $u^{i,p}(x,y)$ are smooth p-forms which are to be determined and $u^{0,p}(x,x)$ is the identity of $\Lambda^p T_x^*(M)$.

Theorem 1. *In order that* (1.2) *satisfies the heat equation* (1.1)*, the* $u^{i,p}(x,y)$ *must satisfy the following system of recursion relations*

$$\left(i + \frac{r}{4g}\frac{dg}{dr} \right) u^{i,p}(x,y) + \nabla_{r\partial_r} u^{i,p}(x,y) - \Delta_y u^{i-1,p}(x,y) = 0, \tag{1.3}$$

for $i = 0, 1, \ldots, N-1$ *and moreover,*

$$\left(\frac{\partial}{\partial t} - \Delta_y \right) H_N^p(t,x,y) = -\frac{e^{-r^2/4t}}{(4\pi t)^{n/2}} t^N \, u^{N,p}(x,y). \tag{1.4}$$

The integer N *is chosen to be larger than* $n/2$*. These conditions determine the double* (p,p) *forms* $u^{i,p}(x,y)$ *uniquely in a sufficiently small neighborhood of the diagonal.*

Proof. Fix an arbitrary point $x \in M$ and introduce normal coordinates in an open neighborhood U of x such that $g_{ij}(x) = \delta_{ij}$ and x has the coordinate representation $(0, \ldots, 0)$. Let $F(r(x,y))$ be a function of r depending only on the geodesic distance of y from x and ω is any C^∞ p-form defined on U. Then the Laplacian with respect to y has the following structure in terms of r:

$$\begin{aligned} \Delta_y(F(r)\omega) = &\left(\frac{d^2F}{dr^2}(r) + \frac{n-1}{r}\frac{dF}{dr}(r) + \frac{1}{2g}\frac{dg}{dr}\frac{dF}{dr}(r) \right)\omega \\ &+ \frac{2}{r}\frac{dF}{dr}(r)\nabla_{r\partial_r}\omega + F(r)\Delta\omega, \end{aligned} \tag{1.5}$$

where $g(y) = \det(g_{ij}(y))$. For the case in which $F(r) = \exp\left(-\frac{r^2}{4t}\right)$, substitution in (1.5) implies that

$$\begin{aligned} &\Delta_y \left(\exp\left(-\frac{r^2}{4t} \right) \omega \right) \\ &= e^{-\frac{r^2}{4t}} \left[\left(\left(\frac{r^2}{4t^2} - \frac{1}{2t} \right) - \frac{n-1}{2t} - \frac{r}{4gt} \right) \omega - \frac{1}{t}\nabla_{r\partial_r}\omega + \Delta\omega \right]. \end{aligned}$$

It is also found that

$$\frac{\partial}{\partial t} H_N^p(t,x,y) = \frac{r^{-r^2/4t}}{(4\pi t)^{n/2}}$$
$$\times \sum_{i=0}^{N} \Big[\left(\frac{r^2}{4t^2} + \frac{1}{t}\left(i - \frac{n}{2}\right) - \frac{r^2}{4t^2} + \frac{1}{2t} + \frac{n-1}{2t} + \frac{r}{4gt}\frac{dg}{dr} \right) t^i u^{i,p}(x,y)$$
$$+ t^{i-1} \nabla_{r\partial_r} u^{i,p}(x,y) - t^i \Delta_y u^{i,p}(x,y) \Big]$$
$$= \frac{e^{-r^2/4t}}{(4\pi t)^{n/2}}$$
$$\times \sum_{i=0}^{N} \left[\left(i + \frac{r}{4g}\frac{dg}{dr} \right) t^{i-1} u^{i,p}(x,y) + t^{i-1} \nabla_{r\partial_r} u^{i,p}(x,y) - t^i \Delta_y u^{i,p}(x,y) \right].$$

When the coefficients of t^{i-1} are equated to zero, (1.3) results and the remaining equation is exactly (1.4). □

It is clear that (1.3) can be put in the equivalent form

$$\nabla_{r\partial_r} u^{i,p}(x,y) + \left(i + \frac{r}{4g}\frac{dg}{dr} \right) u^{i,p}(x,y) = \Delta_y \, u^{i-1,p}(x,y). \tag{1.6}$$

Lemma 1. *For an arbitrary vector field $v \in \Lambda^p T_x^*(M)$, then in the open set U, equations* (1.6) *have unique solutions under the condition $u^{0,p}(v,x) = v$, $u^{-1,p}(x,y) = 0$.*

Proof. System (1.6) can be written in the form

$$\nabla_{r\partial_r} \left(r^i g^{1/4} u^{i,p}(v,y) \right) = r^i g^{1/4} \Delta_y \, u^{i-1,p}(v,y). \tag{1.7}$$

Suppose y is an arbitrary point of U and $w_y(t)$, $0 \le t \le r(x,y)$ the geodesic joining points x and y. The curve $w_y(t)$ defines, with respect to the Riemannian connection, an isomorphism T_{y,t_0} of $\Lambda^p T^*_{w_y(t_0)}(X)$ onto $\Lambda^p T_y^*(X)$, $0 \le t_0 \le r(x,y)$. Let $u^{0,p}(v,y) = g^{-1/4}(y) T_{y,0}(v)$. Then $u^{0,p}(v,x) = v$, and (1.7) is satisfied for $i = 0$. Suppose there is some m so that when $i < m$ the forms $u^{i,p}(v,y)$ have been determined such that they satisfy (1.7). By induction, define $u^{m,p}(v,y)$ to be

$$u^{m,p}(v,y) = \frac{1}{(r(x,y)^m g^{1/4}(y))}$$
$$\times \int_0^r (r(x,u_y(t)))^{m-1} g^{1/4}(u_y(t)) T_{y,t}(\Delta_y u^{m-1,p}(v,u_y(t)))\, dt.$$

This is a C^∞-form and it satisfies (1.7) for $i = m$.

Now (1.4) implies that $i\, u^{i,p}(v,x) = (\Delta_y u^{i-1,p}(v,y))(v,x)$. Thus uniqueness would follow if it is shown that any C^∞ solution ω of $\nabla_{r\partial_r}(\omega) = 0$ satisfying the initial condition vanishes identically. However, this is clear since $\nabla_{r\partial_r}(\omega) = 0$ implies that for all $y \in U$, ω is invariant under parallel displacements along the geodesic joining points x and y. $\square$

Thus, $H_N^p(t, x, y)$ is constructed in a sufficiently small neighborhood of the diagonal in $M \times M$. Let U' be an open neighborhood of the diagonal such that the closure of U' is contained in U. A type of partition of unity can be introduced by taking $\eta(x, y)$ to be a C^∞-function on $M \times M$ such that η is zero outside U' and is one in the neighborhood of the diagonal. Using $\eta(x, y)$, define

$$\begin{aligned} G_N^p(t, x, y) &= \eta(x, y)\, H_N^p(t, x, y), \\ K_N^p(t, x, y) &= \left(\frac{\partial}{\partial t} - \Delta_y\right) G_N^p(t, x, y). \end{aligned} \tag{1.8}$$

For any smooth p-form $\varphi(t, x)$, it is the case that

$$\lim_{t \to 0^+} \int_M G_N^p(t, x, y) \wedge *\varphi(t, x) = \varphi(0, y).$$

For fixed p and N, it is more concise to write $K_N^p(t, x, y)$ as simply $K(t, x, y)$. It is also worth noting the following notation. In the process of constructing the fundamental solution of the operator in (1.2), if M, N_1, N_2 are vector spaces, and there is given an inner product in M, then there is a natural map $\tau\colon (M \times N_1) \otimes (M \times N_2) \to N_1 \otimes N_2$ such that $\tau((m \otimes n_1), (m' \otimes n_2)) = \langle m, m' \rangle\, n_1 \otimes n_2$ for $m, m' \in M$, $n_1 \in N_1$ and $n_2 \in N_2$, $\langle , \rangle$ the inner product on M. The map $\tau(x, y)$ can be denoted simply as (x, y), or even without brackets. Inductively, the following sequence can now be defined

$$\begin{aligned} K^0(t, x, y) &= K(t, x, y), \\ K^m(t, x, y) &= \textstyle\int_0^t ds \int_M (K^{m-1}(s, x, z), K(t - s, z, y))\, dv_z \end{aligned} \tag{1.9}$$

and dv_z is the volume element on M with respect to z.

Since the manifold M is compact, there exist finitely many open sets $V_1, \dots, V_q$ and $U_1, \dots, U_q$ such that $V_r \subset U_r$ where U_r is diffeomorphic to $\mathbb{R}^n$ and $M = \cup V_r$. A partition of unity can be found relative to the open covering $\{V_r\}_1^q$. Suppose η_r are C^∞ functions which take the value one on V_r and have compact supports contained in U_r. If $L(x, y)$ is any double form, define

$$L_{ij}(x, y) = \eta_i(x)\eta_j(y)\, L(x, y).$$

Define P as the set of indices $P = \{i_1 < \cdots < i_p; j_1 < \cdots < j_p\}$ so for a form,

$$L(x, y) = \sum_P L_{i_1,\dots,i_p,j_1,\dots,j_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \otimes dy_{j_1} \wedge \cdots \wedge dy_{j_p}. \tag{1.10}$$

The $L(x,y)$ will have support contained in $U_i \times U_j$. Now define,

$$||L||_{ij} = \sum_P \sup_{x \in U_i, y \in U_j} |L_{i_1,\dots,i_p,j_1,\dots,j_p}|.$$

Theorem 2. *With respect to* (1.9)*, the following bounds obtain,*

$$||K^0_{i,j}||_{ij} \le C_0 t^{N-\frac{n}{2}},$$
$$||K^{m-1}_{i,j}||_{ij} \le (CC_0)^m t^{m(N-\frac{n}{2})+m-1} \frac{(\Gamma(N-\frac{n}{2}+1))^m}{\Gamma(m(N-\frac{n}{2})+m)}, \tag{1.11}$$

where C_0, C *are constants.*

Proof. Using (1.4), we have

$$\begin{aligned} ||K^0_{i,j}||_{ij} &= ||K(t,x,y)\eta_i(x)\eta_j(y)|| \le ||K(t,x,y)||_{i,j} \\ &= \sum_P \sup |K_{i_1,\dots,i_p,j_1,\dots,j_p}| \le C_0 t^{N-\frac{n}{2}}. \end{aligned}$$

Taking the second result of (1.11) as an induction hypothesis, it is the case that

$$\begin{aligned} K^m_{i,j}(t,x,y) &= \int_0^t ds \int_M \eta_i(x) \\ &\quad \times \sum_{r=1}^q \varphi_r(x) K^{m-1}(s,x,z)\eta_j(y)K(t-s,z,y)dv_z \\ &= \sum_{r=1}^q \int_0^t ds \int_M (\eta_i(x)\varphi_r(z)K^{m-1}(s,x,z)\eta_j(y)K(t-s,z,y)\, dv_z. \end{aligned}$$

Employing (1.11), it follows that

$$\begin{aligned} ||K^m_{i,j}(t,x,y)|| &\le C_1(CC_0)^m \\ &\quad \times \frac{(\Gamma(N-\frac{n}{2}+1))^m}{\Gamma(m(N-\frac{n}{2})+m)} \cdot C_0 \int_0^t s^{m(N-\frac{n}{2})+m-1}(t-s)^{N-\frac{n}{2}}\, ds. \end{aligned}$$

Here, C_1 is a constant which is independent of m. By choosing $C > C_1$ and realizing that the integral is of beta function type, the following estimate is obtained,

$$||K^m_{i,j}(t,x,y)|| \le (CC_0)^{m+1} t^{(m+1)(N-\frac{n}{2})+m} \frac{\Gamma(N-\frac{n}{2}+1)^{m+1}}{\Gamma((m+1)(N-\frac{n}{2})+m+1)}.$$

This finishes the proof by induction. □

2. Fundamental Solution of the Heat Operator

At this point we continue in the direction of obtaining a fundamental solution of the heat operator.

$$\begin{aligned} &e^p(t,x,y) \\ &= G_N^p(t,x,y) + \sum_{m\geq 0} (-1)^{m+1} \int_0^t ds \int_M (K^m(s,x,z), G_N^p(t-s,z,y)) dv_z. \end{aligned} \tag{2.1}$$

On account of Theorem 2, the series on the right-hand side of (2.1) converges to a double form and can be considered a C^∞-(p,p) form.

Theorem 3. *The form $e^p(t,x,y)$ is the fundamental solution of the heat operator acting on p-forms.*

Proof. Continuing to write $K_N^p(t,x,y)$ as $K(t,x,y)$, using (1.8) it is found that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_y\right) e^p(t,x,y) &= K(t,x,y) + \left(\frac{\partial}{\partial t} - \Delta_y\right) \\ &\quad \times \sum_{m\geq 0} (-1)^{m+1} \int_0^t ds \int_M (K^m(s,x,z), G_N^p(t-s,z,y)) dv_z \\ &= K(t,x,y) + \sum_{m\geq 0} (-1)^{m+1} (K^m(t,x,y) + K^{m+1}(t,x,y)) \\ &= K(t,x,y) - K(t,x,y) = 0. \end{aligned}$$

□

Theorem 4. *As $t \to 0^+$,*

$$\sum_{m=0}^{\infty} (-1)^{m+1} \int_0^t ds \int_M (K^m(s,x,z), G_N^p(t-s,z,x))\, dv_z = O(t^{N-\frac{n}{2}}). \tag{2.2}$$

Proof. Beginning with Theorem 2, it is the case that

$$\begin{aligned} I &= \sum_{m=0}^{\infty} (-1)^{m+1} \int_0^t ds \int_M (K^m(s,x,z), G_N^p(t-s,z,x))\, dv_z \\ &= O\left[\int_0^t ds \int_0^\infty \exp\left(-\frac{r^2}{4(t-s)}\right) (t-s)^{-\frac{n}{2}} s^{N-\frac{n}{2}} r^{n-1}\, dr\right]. \end{aligned}$$

To simplify this, substitute $r = 2\kappa(t-s)^{1/2}$,

$$I = O\left[\int_0^t ds \int_0^\infty d\kappa\, e^{-\kappa^2}(t-s)^{-\frac{n}{2}} s^{N-\frac{n}{2}} 2^n \kappa^{n-1}(t-s)^{\frac{n}{2}}\right]$$
$$= O\left[\int_0^t s^{N-\frac{n}{2}}\, ds \int_0^\infty \kappa^{n-1} e^{-\kappa^2}\, d\kappa\right]$$
$$= O\left[\int_0^t s^{N-\frac{n}{2}}\, ds\right] = O(t^{N-\frac{n}{2}+1}).$$

□

From the definition of $e^p(t,x,y)$ in (2.1), it is now possible to calculate

$$(\mathrm{Tr} e^p)(t,x,x) = (\mathrm{Tr} G_N^p)(t,x,x) + O(t^{N-\frac{n}{2}+1})$$
$$= (\mathrm{Tr} H_N^p)(t,x,x) + O(t^{N-\frac{n}{2}+1})$$
$$= (4\pi t)^{-\frac{n}{2}} \sum_{i=0}^N t^i \,\mathrm{Tr}\, u^{i,p}(x,x) + O(t^{N-\frac{n}{2}+1}),$$

since $r = 0$ when x coincides with y. The double forms H_N^p satisfy (1.2) as do the $u^{i,p}$. Since $u^{0,p}(x,x)$ is the identity endomorphism of $\Lambda^p T_x^*$, the following theorem has been proved.

Theorem 5.

$$(\mathrm{Tr} e^p)(t,x,x) = (4\pi t)^{-\frac{n}{2}} \left\{ \binom{n}{p} + \sum_{i=1}^N t^i \,\mathrm{Tr}\, u^{i,p}(x,x) \right\} + O(t^{N-\frac{n}{2}+1}). \tag{2.3}$$

Suppose $\omega_n(x)$ is a set of eigenforms for the Laplace operator. Define the expression

$$f_m(t,x) = \int_M e^p(t,x,y)\omega_m(y) dv_y. \tag{2.4}$$

Then it can be observed that

$$\frac{\partial}{\partial t} f_m(t,x) = \int_M \frac{\partial}{\partial t} e^p(t,x,y)\omega_m(y) dv_y = \int_M \Delta_x e^p(t,x,y)\omega_m(y) dv_y$$
$$= \int_M \Delta_y e^p(t,x,y)\omega_m(y) dv_y = \int_M e^p(t,x,y)\Delta_y \omega_m(y) dv_y$$
$$= \lambda_m f_m(t,x).$$

This result implies that

$$f_m(t,x) = \omega_m(x)\, e^{\lambda_m t}.$$

To summarize, the following has been shown.

Theorem 6. *The fundamental solution $e^p(t,x,y)$ has a representation of the form*

$$e^p(t,x,y) = \sum_m \omega_m(x)\omega_m(y)e^{\lambda_m t}. \tag{2.5}$$

Since the right-hand side converges absolutely, the result follows from completeness of the eigenforms.

3. The Gauss-Bonnet Theorem

The quantities $u^{i,p}$ which appear in the expansion (1.1) contain valuable information with regard to the structure of the underlying manifold. Of interest here is the fact that they can be related to the Euler characteristic $\chi(M)$ of the manifold M. Suppose M is a compact, oriented, connected manifold of dimension n. A superscript on the Laplace operator indicates the degree of the form space acted on. To establish this link, some further results are required.

Lemma 2. *For $\lambda \in \mathbb{R}^-$, let E^p_λ be the λ-eigenspace, possibly trivial, for Δ^p. Then the sequence $0 \to E^0_\lambda \overset{d}{\to} \cdots \overset{d}{\to} E^n_\lambda \to 0$ is exact.*

Proof. If $\omega \in E^p_\lambda$ then $\Delta^{p+1}\, d\omega = d\Delta^p\omega = \lambda d\omega$, so $d\omega$ is an eigenform of Δ^p and so $d\omega \in E^{p+1}_\lambda$. The sequence is well defined and has $d^2 = 0$. Suppose $\omega \in E^p_\lambda$ has $d\omega = 0$, then $\lambda\omega = \Delta^p\omega = -(\delta d + d\delta)\omega = -d\delta\omega$. Therefore, we can write

$$\omega = d\left(-\frac{1}{\lambda}\delta\omega\right)$$

since $\lambda \neq 0$. □

Lemma 3. *The operator $D = d + \delta \colon \oplus_k E^{2k}_\lambda \to E^{2k+1}_\lambda$ is an isomorphism, therefore*

$$\sum_p (-1)^p \dim E^p_\lambda = 0.$$

Corollary 1. *Let $\{\lambda^p_i\}$ be the spectrum of Δ^p, then*

$$\sum_p (-1)^p \sum_i e^{\lambda^p_i t} = \sum_p (-1)^p {\sum_i}' e^{\lambda^p_i t}.$$

The second sum runs over those i for which $\lambda^p_i = 0$. Consequently,

$${\sum_i}' e^{\lambda^p_i t} = \dim\ker\Delta^p.$$

As a result of this, it must be that

$$\sum_p (-1)^p \mathrm{Tr}\, e^{t\Delta^p} = \sum_p (-1)^p \sum_i e^{t\lambda^p_i},$$

and is independent of t. This means that the large t behavior of the operator is the same as the small t behavior. The large t behavior of $\mathrm{Tr}\, e^{-t\Delta}$ is related to the de Rham cohomology while the small t permits us to make the identification

$$\begin{aligned}\chi(M) &= \sum_p (-1)^p \dim\, H^p_{dR}(M) \\ &= \sum_p (-1)^p \dim\ker\Delta^p = \sum_p (-1)^p \mathrm{Tr}\, e^{t\Delta^p} \\ &= \sum_p (-1)^p \int_M \mathrm{Tr}\, e^p(t,x,x)\, dv_x. \end{aligned} \tag{3.1}$$

Consequently, using the result for the expansion of $e^p(t,x,,x)$ given in Theorem 5, the following expression is obtained for the Euler characteristic of M,

$$\chi(M) = \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} \left(\int_M \sum_{i=0}^{n} (-1)^i \mathrm{Tr}\, u^{i,k}(x,x)\, dv_x \right) t^k. \tag{3.2}$$

Since $\chi(M)$ is independent of t, only the constant term on the right can be nonzero. Therefore, we have established the following form of the generalized Gauss-Bonnet Theorem.

Theorem 7.

$$(4\pi)^{-\frac{n}{2}} \int_M \sum_{i=0}^{n} (-1)^i\, u^{i,k}(x,x)\, dv_x = \begin{cases} 0, & k \neq \frac{n}{2}, \\ \chi(M), & k = \frac{n}{2},\ n \text{ even}. \end{cases} \tag{3.3}$$

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