ON GENERALIZED SIMULTANEOUS NEAREST POINT IN NORMED SPACES

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Abstract. In this paper, we study the problem of generalized simultaneous approximation in terms of the Minkowski functional. We develop a theory of generalized best simultaneous approximation in quotient spaces and introduce equivalent assertions between the subspaces W and $W+M$ and the quotient space W/M. Some other results regarding generalized simultaneous approximation in Banach space are presented.

1. Introduction

Let G be a subspace of the normed space X and $x \in X$. A point $g_0 \in G$ is said to be a best approximation to x from G, whenever

$$\|x-g_0\| = \inf_{g\in G} \|x-g\| = d\left(x,G\right).$$

The set of all best approximations to x from G is denoted by $P_G\left(x\right)$. The set G is called proximinal if $P_G\left(x\right) \neq \phi$ for all $x \in X-G$, see [1, 4, 9]. If a bounded set A is given in X one might want to approximate all elements of A simultaneously by a single element of G. These type of problems arise when a function being approximated is not known precisely but it is known to belong to a set. Several mathematicians have studied this problem of simultaneous approximation in linear spaces see [5, 8]. A point $g_0 \in G$ is called a best simultaneous approximation to A from G, whenever

$$\sup_{a\in A} \|a-g_0\| = \inf_{g\in G} \sup_{a\in A} \|a-g\| = d\left(A,G\right).$$

The set of all best simultaneous approximations to A from G is denoted by $P_G\left(A\right)$. The set G is called simultaneously proximinal if $P_G\left(A\right) \neq \phi$ for all bounded subsets of $X-G$. By taking the set A to be $\{x\}$, the problem of best simultaneous approximation is considered as a generalization of best approximation.

For a closed bounded convex subset C of X with $0 \in int(C)$, recall that the Minkowski functional $\rho_c\colon X \to \mathbb{R}$ with respect to the set C is defined by

$$\rho_c(x) = \inf\left\{\alpha > 0 : x \in \alpha C\right\}$$

for all $x \in X$. For a closed non-empty subset G of X and $x \in X$, define the generalized distance function by

$$d_c(x, G) = \inf_{g \in G} \rho_c(x - g).$$

A point $g_0 \in G$ with $d_c(x, G) = \rho_c(x - g_0)$ is called a generalized nearest point (generalized best approximation) [13]. For a non-empty bounded subset A of X and a non-empty subset G of X, we define

$$d_c(A, G) = \inf_{g \in G} \sup_{a \in A} \rho_c(a - g).$$

If there exists an element $g_0 \in G$ such that $\sup_{a \in A} \rho_c(a - g_0) = d_c(A, G)$, then g_0 is called a best simultaneous ρ_c-approximation to A from G. The set of all simultaneous ρ_c-approximations to A from G is denoted by

$$P_{G,C}(A) = \left\{ g \in G : d_c(A, G) = \sup_{a \in A} \rho_c(a - g) \right\}.$$

We say that G is simultaneously ρ_c-proximinal if for every bounded subset A of X, $P_{G,C}(A) \neq \phi$ and simultaneously ρ_c-Chebyshev if $P_{G,C}(A)$ is singleton.

In this paper we study the problem of best simultaneous ρ_c-approximation in terms of the Minkowski functional. Some results on quotient space and on simultaneous ρ_c-approximation of the sum of two subspaces are obtained by generalizing some of the results in [5, 12].

Throughout this paper, X is a normed space and C is a closed bounded convex subset of X.

2. P_C-Simultaneous Approximation in quotient Space

In this section we begin with the following proposition.

Proposition 1 ([13, Proposition 2.1])**.**

(1) $\rho_c(x) \geq 0$ *and* $\rho_c(x) = 0 \Leftrightarrow x = 0$.
(2) $\rho_c(x + y) \leq \rho_c(x) + \rho_c(y)$.
(3) $-\rho_c(y - x) \leq \rho_c(x) - \rho_c(y) \leq \rho_c(x - y)$.
(4) $\rho_c(\lambda x) \geq \lambda\ \rho_c(x), \lambda \geq 0$.
(5) $\rho_c(-x) = \rho_{-c}(x)$.
(6) $\rho_c(x) = 1 \Leftrightarrow x \in \partial C$.
(7) $\rho_c(x) < 1 \Leftrightarrow x \in int(C)$.
(8) $\alpha \|x\| \leq \rho_c(x) \leq \beta \|x\|$, *where* $\alpha = \inf_{x \in \partial C} \rho_c(x)$ *and* $\beta = \sup_{a \in C} \rho_c(x)$.

Now we show some properties of the ρ_c-distance function and the set of ρ_c-best simultaneous approximations.

Proposition 2. *Let G be a subspace of X and $A \subseteq X$ be a bounded subset of X. Then:*

(1) *$P_{G,C}(A)$ is bounded.*
(2) *If G is convex then, $P_{G,C}(A)$ is convex.*
(3) *If G is closed then, $P_{G,C}(A)$ is closed.*
(4) *$d_c(A+y, G+y) = d_c(A, G)$.*
(5) *$P_{G+y,C}(A+y) = P_{G,C}(A) + y$.*
(6) *$P_{\lambda G,C}(\lambda A) = |\lambda| P_{G,C}(A)$.*

Proof.
(1) Let $g_0 \in P_{G,C}(A)$. Then for $a \in A$, the inequality

$$\rho_c(g_0) \leq \rho_c(a - g_0) + \rho_c(a),$$

implies that

$$\begin{aligned}\rho_c(g_0) &\leq \sup_{a\in A} \rho_c(a - g_0) + \sup_{a\in A} \rho_c(a)\\ &= d_c(A, G) + \sup_{a\in A} \rho_c(a).\end{aligned}$$

(2) Let $g_1, g_2 \in P_{G,C}(A)$. Then for $0 < \lambda < 1$,

$$\begin{aligned}&\sup_{a\in A} \rho_c(a - (\lambda g_1 + (1-\lambda) g_2))\\ &\leq \lambda \sup_{a\in A} \rho_c(a - g_1) + (1-\lambda) \sup_{a\in A} \rho_c(a - g_2)\\ &= \lambda\, d_c(A, G) + (1-\lambda)\, d_c(A, G) = d_c(A, G).\end{aligned}$$

(3) Let $g_n \in P_{G,C}(A)$ such that $g_n \to g_0$. Then

$$\sup_{a\in A} \rho_c(a - g_n) = d_c(A, G).$$

By continuity of ρ_c, we have $\sup_{a\in A} \rho_c(a - g_0) = d_c(A, G)$. Thus $g_0 \in P_{G,C}(A)$.

(4)

$$\begin{aligned}d_c(A+y, G+y) &= \inf_{g\in G} \sup_{a+y\in(A+y)} \rho_c((a+y) - (g+y))\\ &= \inf_{g\in G} \sup_{a\in A} \rho_c(a - g) = d_c(A, G).\end{aligned}$$

Similarly, we can show that

$$d_{\lambda G}(\lambda A) = |\lambda|\ d_G(A).$$

(5) Let

$$\begin{aligned} & g_0 + y \in P_{G+y,C}(A+y) \\ & \quad \Leftrightarrow d_c\left(A+y, G+y\right) = \sup_{a\in A} \rho_c\left((a+y) - (g_0+y)\right) \\ & \quad = \sup_{a\in A} \rho_c\left(a-g_0\right) \\ & \quad = \inf_{g\in G} \sup_{a\in A} \rho_c\left(a-g\right) \\ & \quad \Leftrightarrow g_0 \in P_{G,C}(A) \\ & \quad \Leftrightarrow g_0 + y \in P_{G,C}(A) + y. \end{aligned}$$

(6) Similarly, we can show that $P_{\lambda G,C}\left(\lambda A\right) = \left|\lambda\right| P_{G,C}(A)$. □

Lemma 3. *Let X be a normed space and M is a ρ_c-proximinal subspace of X. Then for each non-empty bounded set A in X, $d_c\left(A, M\right) = \sup_{a\in A} \inf_{m\in M} \rho_c\left(a-m\right)$.*

Proof. Since M is ρ_c-proximinal, for each $a \in A$, there exists $m_a \in M$ such that

$$\rho_c\left(a - m_a\right) = \inf_{m\in M} \rho_c\left(a-m\right).$$

Now,

$$\begin{aligned} d_c\left(A, M\right) &= \inf_{m\in M} \sup_{a\in A} \rho_c\left(a-m\right) \\ &\leq \sup_{a\in A} \rho_c\left(a-m_a\right) \\ &= \sup_{a\in A} \inf_{m\in M} \rho_c\left(a-m\right) \\ &\leq \inf_{m\in M} \sup_{a\in A} \rho_c\left(a-m\right) \\ &= d_c\left(A, M\right). \end{aligned}$$

This implies $d_c\left(A, M\right) = \sup_{a\in A} \inf_{m\in M} \rho_c\left(a-m\right)$. □

Let M be a subspace of a normed space X. The quotient space X/M is the set of all cosets $x + M$ with the following operations:

(1) $(x+M) + (y+M) = (x+y) + M$.
(2) $\lambda\left(x+M\right) = \lambda x + M$ for every x, y and an arbitrary λ.

For the ρ_c-Minkowski functional on X, we define a function $\widetilde{\rho_c}\colon X/M \to \mathbb{R}$ such that

$$\widetilde{\rho_c}\left(x+M\right) = \inf_{m\in M} \rho_c\left(x+m\right).$$

Theorem 4. *Let M be a ρ_c-proximinal subspace of X, and $M \subseteq W$ be a subspace of X. If A is a bounded set in X and $w_\circ \in P_{W,C}(A)$, then $w_\circ + M \in P_{W/M,C}(A/M)$.*

Proof. Since A is a bounded set in X, A/M is bounded in X/M. In fact,

$$\begin{aligned}\|a+M\| &= \inf_{m\in M} \|a+m\| \\ &\leq \|a\| < \infty.\end{aligned}$$

Assume that $w_\circ \in P_{W,C}(A)$ and that $w_\circ + M \notin P_{W/M,C}(A/M)$. This means, there exists $w' \in W$ such that

$$\begin{aligned}\sup_{a\in A} \widetilde{\rho}_c(a - w' + M) &< \sup_{a\in A} \widetilde{\rho}_c(a - w_0 + M) \\ &= \sup_{a\in A} \inf_{m\in M} \rho_c(a - w_0 + m) \\ &= \inf_{m\in M} \sup_{a\in A} \rho_c(a - w_0 + m) \\ &\leq \sup_{a\in A} \rho_c(a - w_0) = d_c(A, W). \qquad (1)\end{aligned}$$

On the other hand, for each $a \in A$ we have

$$\widetilde{\rho}_c(a - w' + M) = \inf_{m\in M} \rho_c(a - w' - m).$$

It follows that for each $\epsilon > 0$ and $a \in A$ there exists $m_a \in M$ such that

$$\rho_c(a - w' - m_a) \leq \widetilde{\rho}_c(a - w' + M) + \epsilon.$$

Since $w' + m_a \in W$, we conclude that

$$d_c(A, W) \leq \sup_{a\in A} \rho_c(a - (w' + m_a)) \leq \sup_{a\in A} \widetilde{\rho}_c(a - w' + M) + \epsilon.$$

Thus, since ϵ is arbitrary

$$d_c(A, W) \leq \sup_{a\in A} \widetilde{\rho}_c(a - w' + M). \qquad (2)$$

From (1) and (2) we have

$$d_c(A, W) \leq \sup_{a\in A} \rho_c(a - w' + M) < d_c(A, W),$$

which is impossible. □

Proposition 5. *Let M be a ρ_c-proximinal subspace of X, and $W \supset M$ a subspace of X. If A is bounded subset of X such that*

$$w_\circ + M \in P_{W/M,C}(A/M) \text{ and } m_\circ \in P_{M,C}(A - w_\circ)$$

then $w_\circ + m_\circ \in P_{W,C}(A)$.

Proof. Since $m_\circ \in P_{M,C}(A - w_\circ)$, then

$$\begin{aligned}\sup_{a\in A} \rho_c(a - w_\circ - m_\circ) &= \inf_{m\in M} \sup_{a\in A} \rho_c(a - w_\circ - m)\\ &= \sup_{a\in A} \inf_{m\in M} \rho_c(a - w_\circ - m) \quad \text{(using Lemma 3)}\\ &= \sup_{a\in A} \widetilde{\rho_c}(a - w_\circ + M)\\ &\le \sup_{a\in A} \widetilde{\rho_c}(a - w + M) \text{ for all } w \in W\\ &\le \sup_{a\in A} \rho_c(a - w) \text{ for all } w \in W.\end{aligned}$$

Hence,

$$\sup_{a\in A} \rho_c(a - (w_\circ + m_\circ)) \le \sup_{a\in A} \rho_c(a - w), \text{ for all } w \in W.$$

Since $w_\circ + m_\circ \in W$, we conclude that $w_\circ + m_\circ \in P_{W,C}(A)$. □

Theorem 6. *Let M be a ρ_c-proximinal subspace of $X, M \subseteq W$ and W is ρ_c-simultaneously proximinal subspace of X. Then for each bounded subset A of X, we have $\pi\left(P_{W,C}(A)\right) = P_{W/M,C}(A/M)$. Note here π is the canonical map defined by: $\pi\colon X \to X/M$ by $\pi(x) = x + M$.*

Proof. First, note that

$$\pi\left(P_{W,C}(A)\right) \subseteq P_{W/M,C}(A/M)$$

and W/M is ρ_c-simultaneously proximinal. Now let

$$w_0 + M \in P_{W/M,C}(A/M),$$

where $w_0 \in W$. Since M simultaneously proximinal, there exists $m_0 \in M$ such that $m_0 \in P_{M,C}(A - w_0)$. Hence, $w_0 + m_0 \in P_{W,C}(A)$. Therefore, $w_0 + M \in \pi\ (P_{W,C}(A))$. □

Theorem 7. *Let W and M be two subspaces of X. If M is simultaneously ρ_c-proximinal, then the following assertions are equivalent.*

(1) *W/M is simultaneously ρ_c-proximinal in X/M.*
(2) *$W + M$ is simultaneously ρ_c-proximinal in X.*

Proof. (1) ⇒ (2) Let A be an arbitrary bounded set in X. Then we have A/M is bounded set in X/M. Since $(W + M)/M = W/M$ and M are ρ_c-simultaneously proximinal, it follows that there exists $w_\circ + M \in (W + M)/M$ and $m_\circ \in M$ such that

$$w_\circ + M \in P_{(W+M)/M,C}(A/M) \text{ and } m_\circ \in P_{M,C}(A - w_\circ).$$

Hence, $w_\circ + m_\circ \in P_{W+M,C}(A)$. This shows that $W + M$ is ρ_c-simultaneously proximinal in X.

(2) $\Rightarrow$ (1) Since $W+M$ is simultaneously ρ_c-proximinal and $M \subseteq W+M$, then $(W+M)/M = W/M$ is simultaneously ρ_c-proximinal. □

3. Chebyshev and Quasi Chebyshev ρ_c-Simultaneous Approximation.

Definition 8. *A closed subset W of X is called ρ_c-simultaneously quasi-Chebyshev if the set $P_{W,C}(A)$ is non-empty and compact in X for every bounded set A in X.*

Theorem 9. *Let W and M be a subspace of X. If M is ρ_c-simultaneously Chebyshev, then the following assertions are equivalent.*

1. *W/M is ρ_c-simultaneously Chebyshev in X/M.*
2. *$W+M$ is ρ_c-simultaneously Chebyshev in X.*

Proof. (1) $\Rightarrow$ (2) By hypothesis $(W+M)/M = W/M$ is ρ_c-simultaneously Chebyshev. Assume that (2) is not true. Then some bounded subset A of X has two distinct ρ_c-simultaneous best approximation say $\ell_\circ$ and ℓ_1 in $W+M$. Thus, we have $\ell_\circ$ and $\ell_1 \in P_{W+M,C}(A)$. Since $M \subseteq W+M$, it follows that

$$\ell_\circ + M \text{ and } \ell_1 + M \in P_{(W+M)/M,C}(A/M) = P_{W/M,C}(A/M).$$

But W/M is ρ_c-simultaneously Chebyshev, and so $\ell_\circ + M = \ell_1 + M$. Then there exists $m_\circ \in M\backslash\{0\}$ such that $\ell_1 = \ell_\circ + m_\circ$. Thus,

$$\begin{aligned}\sup_{a\in A} \rho_c((a-\ell_\circ)-m_\circ) &= \sup_{a\in A} \rho_c(a-\ell_1)\\ &= \inf_{m\in M} \sup_{a\in A} \rho_c((a-\ell_\circ)-m)\\ &= d_c(A-\ell_0, M).\end{aligned}$$

So this shows that both $m_\circ$ and 0 are ρ_c-simultaneous best approximation to $A-\ell_\circ$ from M. Hence, M is not ρ_c-simultaneously Chebyshev. This is a contradiction.

(2) $\Rightarrow$ (1) Assume that (1) does not hold. Then for some bounded subset A of X, A/M has two distinct ρ_c-simultaneous best approximation, say $w+M$ and $w'+M \in W/M$. Thus, $w-w' \notin M$. Since M is ρ_c-simultaneously proximinal, there exist ρ_c-simultaneous best approximation m and m' to $A-w$ and $A-w'$ from M, respectively. Therefore, we have

$$m \in P_{M,C}(A-w) \text{ and } m' \in P_{M,C}(A-w').$$

Since $M \subseteq W+M$ and

$$w+M, w'+M \in P_{W/M,C}(A/M) = P_{(W+M)/M,C}(A/M),$$

then by Proposition 5, it follows that $w+m$ and $w'+m' \in P_{W+M,C}(A)$. But, $W+M$ is ρ_c-simultaneously Chebyshev. Thus, $w+m=w'+m'$ and $w-w' \in M$. This is a contradiction. □

Theorem 10. *Let M and W be two subspaces of X. If M is finite dimensional, then the following assertions are equivalent.*

(1) *W/X is ρ_c-simultaneously quasi-Chebyshev in X/M.*
(2) *$W+M$ is ρ_c-simultaneously quasi-Chebyshev in X.*

Proof. (1) ⇒ (2) Suppose that $(W+M)/M$ is ρ_c-simultaneously proximinal in X/M. Since M is also ρ_c-simultaneously proximinal, $W+M$ is ρ_c-simultaneously proximinal in X. Let A be an arbitrary bounded set in X. Then $P_{W+M,C}(A) \neq \phi$. Now to show that $P_{W+M,C}(A)$ is compact, we need to show that every sequence in $P_{W+M,C}(A)$ has a convergent subsequence. Let $\{g_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $P_{W+M,C}(A)$. Then for each $n \geq 1$ by Theorem (4), $g_n + M \in P_{(W+M)/M,C}\left(A/M\right)$. Since $P_{(W+M)/M,C}\left(A/M\right)$ is compact, there exists $g_0 \in W+M$ with $g_0 + M \in P_{(W+M)/M,C}\left(A/M\right)$ and a subsequence $\{g_{n_k}+M\}_{k=1}^{\infty}$ of $\{g_n+M\}_{n=1}^{\infty}$ converges to g_0+M. Now, for all $k \geq 1$, we have

$$\begin{aligned}\widetilde{\rho_c}\left(g_0 - g_{n_k} + M\right) &= \inf_{m\in M} \rho_c\left(g_0 - g_{n_k} - m\right)\\ &= d_c\left(g_0 - g_{n_k}, M\right), \text{ for all } k \geq 1.\end{aligned}$$

Since M is ρ_c-proximinal in X, then for each $k \geq 1$, there exists $m_{n_k} \in M$ such that $m_{n_k} \in P_{M,C}\left(g_0 - g_{n_k}\right)$, and hence,

$$\rho_c\left(g_0 - g_{n_k} - m_{n_k}\right) = d_c\left(g_0 - g_{n_k}, M\right).$$

Therefore, $\lim\limits_{k\to\infty} \rho_c\left(g_0 - g_{n_k} - m_{n_k}\right) = 0$.

On the other hand, $\{g_{n_k}\}_{k=1}^{\infty}$ is a bounded sequence because

$$g_n \in P_{W+M,C}(A).$$

Thus, we have $\{m_{n_k}\}_{k=1}^{\infty}$ is a bounded sequence in M. Moreover, M is a finite dimensional subspace of X. It follows that $\{m_{n_k}\}_{k=1}^{\infty}$ converges to an element $m_0 \in M$. Let $g' = g_0 - m_0$. Then $g' \in W+M$ and using Proposition 1 part 8 we have

$$\rho_c\left(m_{n_k} - m_0\right) \leq \beta \left\|m_{n_k} - m_0\right\| \to 0.$$

This implies

$$\rho_c\left(m_{n_k} - m_0\right) \to 0.$$

Consequently;

$$\begin{aligned}\rho_c\left(g' - g_{n_k}\right) &= \rho_c\left(g_0 - m_0 - g_{n_k}\right)\\ &\leq \rho_c\left(g_0 - g_{n_k} - m_{n_k}\right) + \rho_c\left(m_{n_k} - m_0\right), \text{ for all } k \geq 1.\end{aligned}$$

Thus,

$$\underset{k\to\infty}{lim}\,\rho_c\left(g'-g_{n_k}\right)=0.$$

Since $\{g_{n_k}\}_{k=1}^{\infty}\in P_{W+M,C}(A),$ for all $k\geq 1$, and $P_{W+M,C}(A)$ is closed, we conclude that $g'\in P_{W+M,C}(A)$. Hence, $P_{W+M,C}(A)$ is compact.

(2) $\Rightarrow$ (1) Since M and $W+M$ are ρ_c-simultaneously proximinal and $M\subseteq W+M$, then $(W+M)/M=W/M$ is ρ_c-simultaneously proximinal in X/M.

Now, let A be an arbitrary bounded set in X. Then $P_{W/M,C}(A/M)$ is non-empty. So from the hypothesis we have $W+M$ is ρ_c-simultaneously quasi Chebyshev in X, and hence $P_{W+M,C}(A)$ is compact in X. But we have

$$P_{(W+M)/M,C}(A/M)=\pi\left(P_{W+M,C}(A)\right).$$

It follows that $P_{W/M,C}(A/M)$ is compact. Therefore, W/M is ρ_c-simultaneously quasi Chebyshev in X. □

4. Conclusion

The study of minimal time function $d_c(\cdot,G)$, $d_c(x,G)=\inf\limits_{g\in G}\rho_c(x-g)$ is motivated by its own worldwide applications in many areas of variation analysis, control theory, approximation theory, etc., and has received a lot of attention, see e.g. [13, 3, 6, 7].

In this paper, our interest is to focus on the following minimization problem:

$$\min_{g\in G}\max_{a\in A}\rho_c\left(a-g\right),$$

where A is a bounded set in X. Any solution to this problem $\min\max(A,G)$ is called a best simultaneous approximation or (generalized best simultaneous approximation) to A from G.

We proved our results using the same techniques of the author in [5, 9, 12] but replacing the norm by the Minkowski functional ρ_c. We define the function $\widetilde{\rho_c}\colon X/M\to\mathbb{R}$, by $\widetilde{\rho_c}(x+M)=\inf\limits_{m\in M}\rho_c(x+m)$. We use this function in Theorems 4, 6, 7, 9, and 10 replacing the norm of the coset $x+M$ in X/M.

Note that in the special case when C is the closed unit ball, the minimal time function $d_c(x,G)$ and the corresponding minimization problem $\min\limits_{g\in G}\rho_c(x-g)$ are reduced to the distance function $d(x,G)=\inf\limits_{g\in G}\|x=g\|$ and to the classical best approximation. This means, the existence results in [5, 9, 12] is a special case of our results in the sense that the set C is taken to be the closed unit ball.

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