LATTICE PROPERTIES OF T_1-L TOPOLOGIES

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Abstract. We study the lattice structure of the set $\Omega(X)$ of all T_1-L topologies on a given set X. It is proved that $\Omega(X)$ has dual atoms (anti atoms) if and only if membership lattice L has dual atoms (anti atoms). Some other properties of this lattice are also discussed.

1. Introduction

The purpose of this note is to investigate the lattice structure of the collection of all T_1-L topologies. In [5], Johnson studied the lattice structure of the set of all L-topologies on a given set X. It is quite natural to find sublattices in the lattice of L-topologies and study their properties. The collection of all T_1-L topologies on a given set X forms one of the sublattice of the lattice of L-topologies on X. One distinguishing feature between these two lattices is that the lattice of L-topologies is atomic while the collection of all T_1-L topologies is not. Lattice of T_1-L topologies is a complete sublattice of lattice of L-topologies. Also, the collection of all T_1-L topologies is neither modular nor atomic. In [8] Liu determined dual atoms in the lattice of T_1 topologies and Frolich [2] proved this lattice is dually atomic. However, we prove that the collection of all T_1-L topologies has dual atoms if and only if L has dual atoms and that the collection of all T_1-L topologies is not dually atomic.

2. Preliminaries

It is assumed that the reader is familiar with the definitions of lattice, sublattice, complemented lattice, complete lattice, modular lattice, infimum and supremum, atom [4] and L-topology [1]. Dual atom will refer to the notion of dual to atom.

Definition 2.1. [10] *A fuzzy point x_λ in a set X is a fuzzy set in X defined by*

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x, \\ 0 & \text{if } y \neq x; \end{cases} \quad \text{where } 0 < \lambda \leq 1.$$

Definition 2.2. [10] *An L-topological space (X, F) is said to be a T_1-L topology if for every two distinct fuzzy points x_p and y_q, with distinct support, there exists an $f \in F$ such that $x_p \in f$ and $y_q \notin f$ and another $g \in F$ such that $y_q \in g$ and $x_p \notin g$ for all $p, q \in L \backslash \{0\}$.*

Remark 2.3. *For an arbitrary fuzzy point x_λ we allow $0 < \lambda \leq 1$ so as to include all crisp singletons. Hence, every crisp T_1-topology is a T_1-L topology by identifying it with its characteristic function. If τ is any topology on a finite set then τ is T_1 if and only if it is discrete. However, the same is not true in L-topology.*

For a lattice L, recall from [3] *that an element $p \in L$, $p \neq 1$, is called prime if $a, b \in L$ with $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $Pr(L)$* [11]*. The scott topology on L is the topology generated by the sets of the form $\{t \in L : t \not\leq p\}$, where $p \in Pr(L)$. Let (X, τ) be a topological space and $f \colon (X, \tau) \to L$ be a function where L has its scott topology. We say that f is scott continuous if for every $p \in pr(L)$, $f^{-1}\{t \in L : t \not\leq p\} \in \tau$. When $L = [0, 1]$, the scott topology coincides with the topology of topologically generated spaces of Lowen* [9].

The set $\omega_L(\tau) = \{f \in L^X ; f \colon (X, \tau) \to L$ is scott continuous $\}$ is an L-topology. An L-topology F on X is called an induced L-topology if there exists a topology τ on X such that $F = \omega_L(\tau)$. If τ is a T_1 topology, $\omega_L(\tau)$ is a T_1-L topology [4]*. A lattice L is modular if and only if, it has no sublattice isomorphic to N_5, where N_5 is a standard non-modular lattice.*

3. Lattice of T_1-L Topologies

For any set X, the set $\Omega(X)$ of all T_1-L topologies on X forms a lattice with natural order of set inclusion. The least upper bound of a collection of T_1-L topologies belonging to $\Omega(X)$ is the T_1-L topology generated by their union and the greatest lower bound is their intersection. The smallest T_1-L topology is the cofinite topology denoted by 0 and largest T_1-L topology is the discrete L topology denoted by 1.

Theorem 3.1. *The lattice $\Omega(X)$ is complete.*

Proof. Let S be a subset of $\Omega(X)$ and

$$G = \bigcap_{\delta \in S} \delta.$$

Then G is a T_1-L topology and G is the greatest lower bound of S. Since any join (resp. meet) complete lattice with a smallest (resp. largest) element is complete, $\Omega(X)$ is complete. □

Note 3.2. *Let CFT denote the crisp cofinite topology, where*

$$CFT = \{\chi_A / A \text{ is a subset of } X \text{ whose complement is finite } \} \cup \{\underline{0}\},$$

χ_A is the characteristic function of A.

Theorem 3.3. *$\Omega(X)$ is not atomic.*

Proof. Atoms in $\Omega(X)$ are the T_1-L topologies generated by $CFT \cup \{x_\lambda\}$, $0 < \lambda \leq 1$, where x_λ is a fuzzy point. Let

$$\mathcal{C} = \{f \in L^X : f(x) > 0 \text{ for all but finite number of points of } X\} \cup \{\underline{0}\}.$$

Then $\mathcal{C}$ is a T_1-L topology and $\mathcal{C}$ cannot be expressed as join of atoms. Hence, $\Omega(X)$ is not atomic. □

Theorem 3.4. *$\Omega(X)$ is not modular.*

Proof. Let $x_1, x_2, x_3 \in X$ and $\alpha, \beta, \gamma \in (0, 1)$.
Let F be the T_1-L topology generated by $CFT \cup \{f_1, f_2, f_3\}$ where f_1, f_2, f_3 are L subsets defined by

$$f_1(y) = \begin{cases} \alpha & \text{when } y = x_1 \\ 0 & \text{when } y \neq x_1 \end{cases}$$

$$f_2(y) = \begin{cases} \alpha & \text{when } y = x_1 \\ \beta & \text{when } y = x_2 \\ \gamma & \text{when } y = x_3 \\ 0 & \text{when } y \neq x_1, x_2, x_3 \end{cases}$$

$$f_3(y) = \begin{cases} \beta & \text{when } y = x_2 \\ \gamma & \text{when } y = x_3 \\ 0 & \text{when } y \neq x_2, x_3. \end{cases}$$

Let F_1 be the T_1-L topology generated by $CFT \cup \{f_1\}$.
Let F_2 be the T_1-L topology generated by $CFT \cup \{f_1, f_2\}$.
Let F_3 be the T_1-L topology generated by $CFT \cup \{f_3\}$.
Then,we notice that
$F_2 \vee F_3 = F$ and
$F_1 \vee F_3 = F$ so that $\{CFT, F_1, F_2, F_3, F\}$ forms a sublattice of $\Omega(X)$ isomorphic to N_5, where N_5 is the standard nonmodular lattice. Hence, $\Omega(X)$ is not modular. □

Theorem 3.5. *$\Omega(X)$ is not complemented.*

Proof. Let F be the T_1-L topology generated by $CFT \cup \{x_\lambda\}$. Then 1 is not a complement of F since $F \wedge 1 \neq 0$. Let H be any T_1-L topology other than 1, the discrete L topology. If $F \subset H$, then H cannot be the complement of F. Suppose that $F \not\subseteq H$, H cannot contain simultaneously all characteristic functions of open sets in τ and all constant L-subsets. The set $K = \{k : k \text{ is a function from } (X, \tau) \text{ to } L \text{ and } k \notin H\}$ is non empty.

Let $F \vee H = G$ and G has the subbasis $\{f \wedge h / f \in F, h \in H\}$. Then G cannot be equal to the discrete L-topology, since there exists at least one subset of K which is not contained in G. Hence, H is not a complement of F. □

Theorem 3.6. *If L has dual atoms, then $\Omega(X)$ has dual atoms.*

Proof. Let τ be a dual atom in the lattice of T_1 topologies. The only topology finer than τ is the discrete topology. Then there exist a subset A of X such that the simple expansion of τ by A is the discrete topology. The characteristic function χ_A of the subset A does not belong to $\omega_L(\tau)$. If α is a dual atom in L, then the T_1-L topology generated by $\omega_L(\tau) \cup \chi_A^\alpha$ is a dual atom in $\Omega(X)$ where

$$\chi_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

□

Theorem 3.7. *If L has no dual atoms, then $\Omega(X)$ has no dual atoms.*

Proof. Let F be any T_1-L topology other than 1, the discrete L-topology. We claim that there exists at least one T_1-L topology finer than F. Since F is a T_1-L topology different from discrete L-topology, F cannot contain at the same time all constant L subsets and all characteristic functions of subsets of X. Since L has no dual atoms, the collection S of L subsets not belonging to F is infinite. If $g \in S$, then $F(g)$, the simple expansion of the T_1-L topology F by g is a T_1-L topology. Thus for any T_1-L topology F there exists a T_1-L topology $G = F(g)$ such that $F \subset G \neq 1$, since S is infinite and g is one of the selected element. Hence, the proof of the theorem is complete. □

Comparing Theorems 3.6 and 3.7, we have the following results.

Theorem 3.8. *The lattice of T_1-L topologies $\Omega(X)$ has dual atoms if and only if L has dual atoms.*

Theorem 3.9. $\Omega(X)$ *is not dually atomic in general.*

Proof. This follows from Theorem 3.7. □

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