WEAK APPROACH TO PLANAR SOAP BUBBLE CLUSTERS

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Abstract. The planar soap bubble problem seeks the least perimeter way to enclose and separate m regions of m given areas. We discuss a useful approach, especially for $m \leq 8$.

1. Introduction

The soap bubble problem in $\mathbb{R}^n$ is the search for the optimal way to enclose m regions of given volumes. For the planar case, for given $A_1, \ldots, A_m > 0$, we seek a length minimizing enclosure of regions $R_1, \ldots, R_m$ of areas $A_1, \ldots, A_m$. The exterior region is denoted by $R_0 = \mathbb{R}^2 \setminus \overline{R_1 \cup \cdots \cup R_m}$. Each region is a union of disjoint open components. Although it is intuitive that each region should be connected, it is hard to prove. For a single area, a circle was proven to be the shortest in 1880. For two areas, in 1991, Foisy et al. [12] proved that the standard double bubble in Figure 1, is uniquely minimizing.

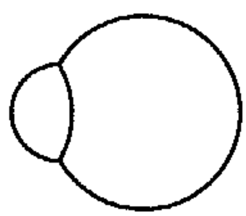

Figure 1. A standard double bubble.

For three areas, in 2002, Wichiramala [23, 24] proved that the standard triple bubble in Figure 2, is the unique minimizer.

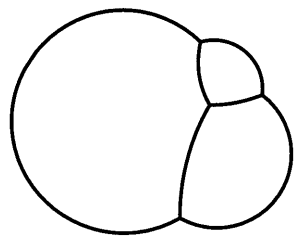

Figure 2. A standard triple bubble.

The problem is still open for $m \geq 4$. In higher dimensions, the standard double bubble was proven to be the best in 2008 [18]. The case of $\mathbb{R}^3$ and $m = 3$ is virtually untouched.

The soap bubble problem is also studied on other surfaces. For example, the double bubble conjecture was settled on a half plane [13, 15], corners and cones [15], flat tori [9, 2], spheres and hyperbolic spaces [16, 7, 10, 8].

As mentioned earlier, the most difficult step is showing that every region, including the exterior one, must be connected. The weak approach was first introduced in [12] to make R_0 connected by allowing $R_1, \ldots, R_m$ to have greater areas than $A_1, \ldots, A_m$. In particular, bounded components of R_0 can be removed as follows. Make each component part of an adjacent component of bounded region and then remove the redundant edge between them. This process yields an enclosure of less length and greater areas. Consequently, we may ignore enclosures with disconnected R_0 in finding the shortest enclosure with areas at least $A_1, \ldots, A_m$. Finally, we have to show that enclosures of greater areas are not minimizing.

Besides the plane, this approach has been used on cones, a half plane, corners, and other scaling-invariant spaces.

In this work, we provide more advantages of using the weak approach. We also show that this approach can make it easier to prove each region connected for $m = 7$ and 8.

2. Previous Results

In this section we list previously known results, focusing on the planar case.

From the argument in [17, 11], we obtain the following existence and regularity theorem.

Theorem 2.1. *For given $A_1, \ldots, A_m > 0$, there exists a minimizing enclosure of areas $A_1, \ldots, A_m$. Each minimizer is composed of finitely many circular or straight edges separating pairs of different regions. These edges* *meet in threes at* 120 *degree angles. There are real numbers* $p_1, \ldots, p_m$, *called* pressures, *such that each edge between* R_i *and* R_j *has curvature* $|p_i - p_j|$ *and curves into the lower pressure region where* p_0 *is set to be* 0*.*

From the main theorem of [4] and Lemma 6 of [5], we obtain the following theorem.

Theorem 2.2. [4, 5] *For a minimizing enclosure, 1) all edges form a connected graph and 2) every two components may meet at most once.*

We define a *bubble* to be an enclosure with properties in Theorems 2.1 and 2.2. An m-bubble is a bubble enclosing m regions. Hence, for $m \geq 3$, an m-bubble has no 2-sided component. A bubble is called *standard* if its regions are connected and, if $m \leq n+1$, has structure described in [3].

In coming figures, the number in $1, 2, \ldots, m$, 0 on each component will indicate the region this component contribute area to.

A variation of a bubble B is a smooth deformation of B by enclosures $\{B_t | 0 \leq t \leq T\}$. From Proposition 4.2 of [11], we have the following lemma.

Lemma 2.3. [11] *Let* B_t *be a variation of a bubble* B*. Then*

$$\frac{d}{dt} l(B_t)\big|_{t=0} = \sum_i p_i \frac{d}{dt} A(R_i^t)\big|_{t=0}.$$

For a component, we give the direction of each edge counterclockwise. The signed turning angle of each edge is then well-defined as follows. If a directed edge is turning left, then its signed turning angle is positive. If it is turning right, then the sign is negative. From the Gauss-Bonnet Theorem, we obtain the next lemma.

Lemma 2.4. *Let* t_i *be the signed turning angles of edges of an* n*-sided component. Then* $\sum_{i=1}^n t_i = \frac{6-n}{3}\pi$*.*

3. The Weak Approach

Although, the weak approach has analogs in higher dimensions, we will focus on the planar case.

We define a weak enclosure for areas $A_1, \ldots, A_m$ to be an enclosure of area $a_1, \ldots, a_m$ where $a_i \geq A_i$. Next, we will show existence of minimizing weak enclosures and list their properties.

Let $L(A_1, \ldots, A_m)$ be the length of a minimizing bubble of areas A_1, …, A_m. Since $L(A_1, \ldots, A_m)$ is continuous and tends to infinity as each A_i approaches infinity, together with compactness argument, we have that

$$\min_{a_i \geq A_i} L(A_1, \ldots, A_m)$$

exists. Equivalently a minimizing weak enclosure exists and necessarily has to minimize the areas it encloses. Hence these minimizers are also bubbles. Consequently, we may call them *minimizing weak bubbles* or *weakly minimizing bubbles*. From Proposition 3.5 of [24] ([23, Proposition 3.6]), we have the following lemma.

Lemma 3.1. [23, 24] *A weak minimizer for areas* $A_1, \ldots, A_m$ *has connected* R_0 *and nonnegative pressures. Moreover, if* $p_i > 0$*, then* $A(R_i) = A_i$*.*

Then we conclude in Theorem 3.6 of [24] ([23, Theorem 3.8]) that the weak approach is valid for $m \leq 6$ as follows.

Proposition 3.2. [23, 24] *For* $m \leq 6$*, if every weakly minimizing* m*-bubble is standard, then every minimizing* m*-bubble is standard.*

Moreover, the weak approach is valid in higher dimensions as in Theorem 3.7 of [24] ([23, Theorem 3.10]).

Proposition 3.3. [23, 24] *In* $\mathbb{R}^n$*, for* $m \leq n+1$*, if every weakly minimizing* m*-bubble is standard, then every minimizing* m*-bubble is standard.*

Now we will generalize Proposition 3.2 to more properties of minimizers. We first conclude from the proof of Proposition 3.6 of [24] ([23, Proposition 3.8]) into smaller steps as the following lemmas.

Lemma 3.4. *For a bubble with nonnegative pressures, if a component has at most five sides, then its pressure is positive.*

Lemma 3.5. *For a weak minimizer, if every component has at most five sides, then all pressures are positive.*

Lemma 3.6. *For* $m \leq 6$*, every* m*-bubble with connected regions and nonnegative pressures must have all pressures positive.*

Lemma 3.7. *For* $m \leq 6$*, every weakly minimizing* m*-bubble with connected regions is also minimizing for the same areas.*

The next lemma indicates that some properties can be inherited from weak minimizers to minimizers.

Lemma 3.8. *Consider minimizers and weak minimizers for areas* $A_1, \ldots,$ A_m*. Suppose 1) every weak minimizer has property P and 2) every weak minimizer with property P has the same length as a minimizer. Then every minimizer has property P.*

Proof. Suppose B is a minimizer and W is a weak minimizer. Hence, from assumption 1), W has property P. Thus, from assumption 2), $l(W) = l(B)$. Consequently, B is a weak minimizer. Finally, from assumption 1), B has property P. □

Again, using the previous two lemmas, we may obtain Proposition 3.2.

The advantage of using the weak approach is that we only have to show that a weak minimizer has the property we desire. Then we can easily conclude that a minimizer also has that desired property. In order to remove an unwanted bubble, we may find a shorter enclosure that encloses greater areas.

In another space, we may use the weak approach to solve the bubble conjecture if we have the following conditions. First a weakly minimizing enclosure must exist. Next we need that every weak minimizer with connected regions must have the same length as a minimizer. With these two conditions, we may prove the conjecture by just showing that every weak minimizer has connected regions. In general, we may replace having connected regions by another property, for example, having combinatorial type in some specific collection.

4. Basic Results

In this section, we list some lemmas needed in the main section. An additional advantage of using the weak approach is also listed here.

For a bubble, we define n_i to be the number of sides of R_i.

Lemma 4.1. *For an m-bubble with connected regions, $\sum_{i=0}^{m} n_i = 6(m-1)$.*

Proof. Let V, E, F be the number of vertices, edges, and faces, respectively. Hence, $F = m + 1$. Since all edges meet in threes, $2E = 3V$. By Euler's Formula, $F - E/3 = F - E + V = 2$ and thus $E = 3(m - 1)$. Since $2E = \sum_{i=0}^{m} n_i$, we have $\sum_{i=0}^{m} n_i = 6(m - 1)$ as desired. □

Hence the total number of edges in an m-bubble with connected regions is $\frac{1}{2}\sum_{i=0}^{m} n_i = 3(m-1)$. In general, for a bubble with possible disconnected regions, we have $\frac{1}{6}\sum_{i=0}^{m} n_i + 2$ components, including those of R_0.

The next lemma ([24, Lemma 5.9] and [23, Lemma 5.11]) shows that a 3-sided component of a bubble can be *reduced* to get another bubble.

Lemma 4.2. [23, 24] *Let B be a bubble with a 3-sided component C. If the three incident edges of C are prolonged into C, they will meet at a point at 120 degree angles and then, after removing all edges of C, create another bubble with the pressures of the remaining regions unchanged.*

Proof. Since reducing C maintains curvatures of every edge, all pressure differences are preserved. Then so are the pressures of remaining regions. It is clear that the remaining edges still form a connected graph and every two components still meet at most one. Therefore the new cluster is a bubble. □

Now we mention some extra advantage of using the weak approach. The main work required is to show that every unwanted bubble is not weakly minimizing. Note that a nonminimizing bubble is not weakly minimizing. Hence one way to show that an unwanted bubble is not weakly minimizing is to show that it is not minimizing by finding an area-preserving variation that preserves length in first order but 1) decreases length in second order or 2) preserves length in second order but causes some conflict mentioned, for example, in [23, 24, 6]. However, in the weak approach, as we allow regions to have greater areas, we need not preserve the areas. More specifically, we may find a variation that increases areas of regions with zero pressures. Then the variation will automatically preserve length in first order. Specifically, if we have one region with zero pressure, we need not preserve the area of the region. But we have to choose the direction of the variation so that we increase the area of that region. In the same fashion, if there are two regions with zero pressures, we need to make sure that both areas are initially increasing or fixed.

5. Main Results

Now we conclude the main results for $m = 7$ and 8.

The argument in this section depends heavily on considering a bubble as a connected graph formed by its edges. This will help reduce the complications in listing all possibilities of bubbles. We recall that a *tree* is a connected graph of degrees 3 or 1 with no closed path. A vertex of degree 1, possibly together with its single edge, is called a *leaf* (see more information in [22]). When a vertex of a tree is adjacent to two vertices of degree 1, its three adjacent edges form a triple junction called a *Y-shape end*. We say an edge is *external* if it is between a component and R_0. A component is *external* if it meets R_0.

First we obtain a result about the number of Y-shape ends in a big tree.

Lemma 5.1. *A tree with at least seven edges has at least two Y-shape ends.*

Proof. First note that a tree has an even number of edges. When a tree has seven edges, its shape is unique and there is exactly two Y-shape ends. For a given tree of at least seven edges, we may make a bigger tree with two more edges by adding two adjacent leaves. The number of Y-shape ends may increase by one or stay the same. Any tree may be obtained by adding pairs of adjacent leaves to the tree of seven edges. Therefore the statement is clear. □

From Lemmas 3.6 and 4.2, we have the following lemma.

Lemma 5.2. *If a 7-bubble with connected regions and nonnegative pressures has a 3-sided bounded component, then all pressures are positive.*

Lemma 5.3. *A 7-bubble with connected regions and nonnegative pressures must have positive pressures.*

Proof. Suppose the contrary, that is, $p_1 = 0$. By Lemma 2.4, R_1 has at least six sides. We will divide the proof into cases according to the number of sides of R_1 and the surrounding regions. Note that R_1 has at most seven sides.
Case 1: R_1 is 6-sided and surrounded by $R_2, R_3, \ldots, R_7$ as in Figure 3.

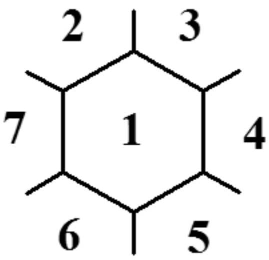

Figure 3. R_1 is surrounded by $R_2, \ldots, R_7$.

By Lemma 2.4, R_1 has straight sides. Hence, $p_2, \ldots, p_7 = 0$ and thus $n_2, \ldots, n_7 \geq 6$. By Lemma 4.1, $36 = 6(m-1) = n_0 + \cdots + n_7 \geq 3+6+6\cdot 6 = 45$, a contradiction. (In addition, we have $p_2, \ldots, p_7 = p_1 = 0$ and this contradicts $l = 2\sum_{i=1}^{m} p_i A_i$ from Proposition 4.3 in [11].)
Case 2: R_1 is 6-sided and surrounded by $R_2, \ldots, R_6, R_0$ as in Figure 4.

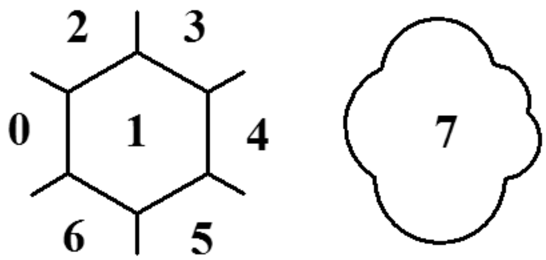

Figure 4. R_1 is surrounded by $R_2, \ldots, R_6, R_0$.

By Lemma 2.4, R_1 has straight sides. By Lemma 4.1, $36 = 6(m-1) \geq 3+6+5\cdot 6+3 = 42$, a contradiction. (In addition, R_7 has an external edge

of turning angle at least 3π, a contradiction.)
Case 3: R_1 is 7-sided. Hence R_1 is surrounded by $R_2, \ldots, R_7, R_0$. Consider the connected graph outside the dotted path (to infinity) in Figure 5.

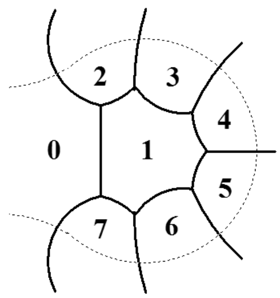

Figure 5. R_1 is surrounded by $R_2, \ldots, R_7, R_0$.

Since the graph has no other face, it is a tree of seven leaves. Hence, by Lemma 5.1, it has at least two Y-shape ends. One Y-shape end may cause R_0 to be 3-sided. The other end causes $R_2, \ldots, R_6$ or R_7 to be 3-sided. By Lemma 5.2, $p_1 > 0$. □

Proposition 5.4. *A weakly minimizing 7-bubble for areas $A_1, \ldots, A_7$ with connected regions is also minimizing for areas A_i.*

Proof. By the previous lemma, all pressures are positive. By Lemma 3.1, $A(R_i) = A_i$ for all i. □

From Lemmas 4.2 and 5.3, we have the following lemma.

Lemma 5.5. *If an 8-bubble with connected regions and nonnegative pressures has a 3-sided bounded component, then all pressures are positive.*

Let 8f be the type in Figure 6.

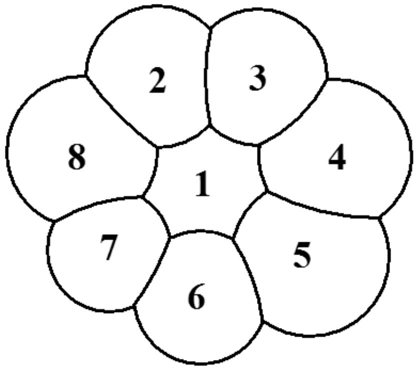

Figure 6. A flower type 8f of 8-bubbles.

Note that the figure looks like a fully blossomed flower. Let 8m be the type in Figure 7.

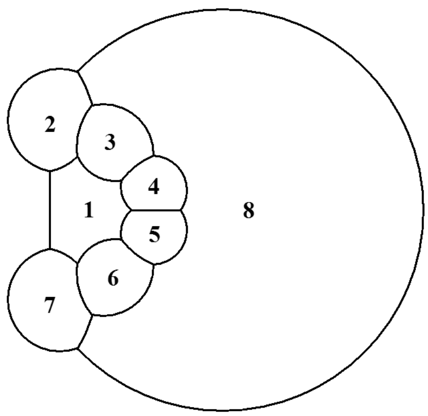

Figure 7. A mangosteen type 8m of 8-bubbles.

Note that the figure looks like a mangosteen, the queen of fruits from Thailand, and has combinatorial type with many symmetries.

Lemma 5.6. *Let B be an 8-bubble with connected regions and nonnegative pressures. Suppose B is not of type 8f with $p_1 = 0$ and is not of type 8m with $p_1 = 0$ or $p_8 = 0$. Then all pressures are positive.*

Proof. First, we relabel regions and suppose the contrary, that $p_1 = 0$. By Lemma 2.4, R_1 has at least six sides. We will divide the proof into cases

according to the number of sides of R_1 and the surrounding regions. Note that R_1 has at most eight sides. Similar to the case $m = 7$, R_1 has at least six sides and at most eight sides.
Case 1: R_1 is 6-sided and surrounded by $R_2, \dots, R_7$ as in Figure 8.

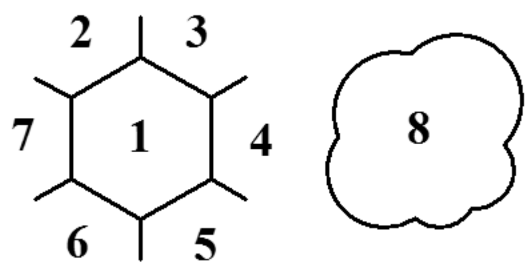

Figure 8. R_1 is surrounded by $R_2, \dots, R_7$.

By Lemma 2.4, R_1 has straight sides. Hence, $p_2, \dots, p_7 = p_1 = 0$. Thus, $n_2, \dots, n_7 \geq 6$. By Lemma 4.1, $42 = 6(m-1) = n_0 + \cdots + n_8 \geq 3 + 6 + 6 \cdot 6 + 3 = 48$, a contradiction.
Case 2: R_1 is 6-sided and surrounded by $R_2, \dots, R_6, R_0$ as in Figure 9.

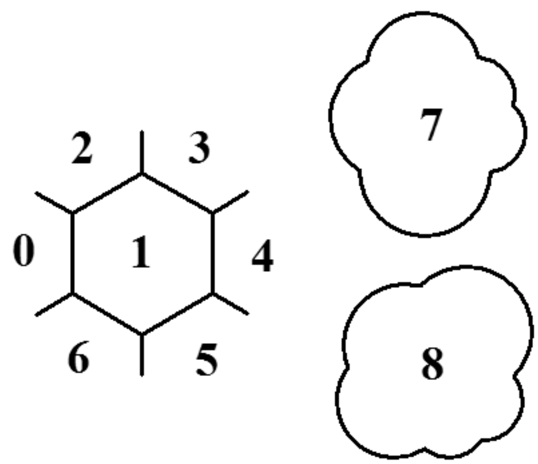

Figure 9. R_1 is surrounded by $R_2, \dots, R_6, R_0$.

Hence, $p_2, \ldots, p_6 = 0$. Thus, $n_2, \ldots, n_6 \geq 6$. By Lemma 4.1, $42 = 6(m-1) = n_0 + \cdots + n_8 \geq 3 + 6 + 5 \cdot 6 + 3 + 3 = 45$, a contradiction.
Case 3: R_1 is 7-sided and surrounded by $R_2, \ldots, R_8$ as in Figure 10.

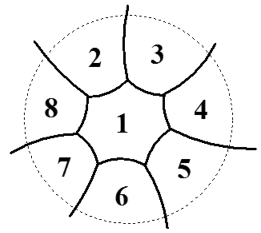

Figure 10. R_1 is surrounded by $R_2, \ldots, R_8$.

Consider the connected graph outside the dotted closed path. Since the graph has only one external face, it is a cycle of external edges decorated by trees inside with seven leaves in total.
Subcase 3.1: The graph has no Y-shape end. Then it is a closed path decorated by seven leaves. Hence B is of the type 8f. By the assumption, $p_1 > 0$.
Subcase 3.2: The graph has a Y-shape end. Hence that Y-shape end causes $R_2, \ldots, R_7$ or R_8 to be 3-sided, a contradiction to Lemma 5.5.
Case 4: R_1 is 7-sided and surrounded by $R_2, \ldots, R_7, R_0$ as in Figure 11.

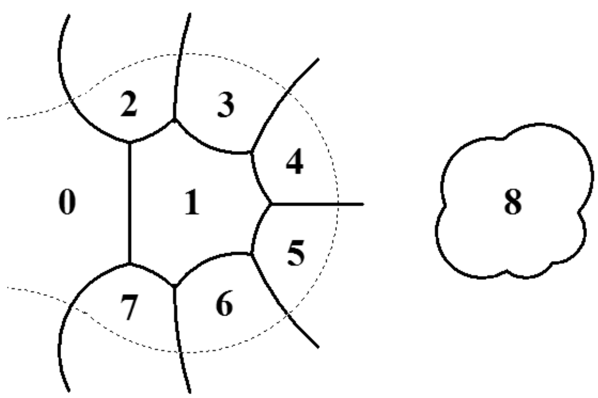

Figure 11. R_1 is surrounded by $R_2, \ldots, R_7, R_0$.

Consider the connected graph outside the dotted path (to infinity). Since the graph has only one face surrounded by the edges of R_8, it is a closed path decorated by trees outside with a total of seven leaves.
Subcase 4.1: The graph has no Y-shape end. Then the graph is a closed path decorated by seven leaves. Hence B is of type 8m. By the assumption, $p_1 > 0$.
Subcase 4.2: The graph has a Y-shape end. If any Y-shape end meets the dotted line, $R_2, \ldots, R_6$ or R_7 becomes 3-sided. Hence one Y-shape end meets R_0, B as in Figure 12.

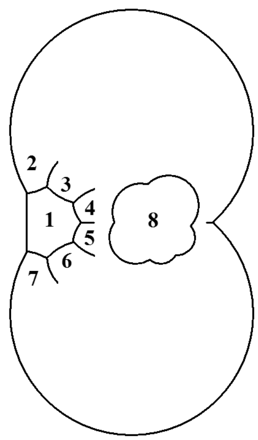

Figure 12. R_1 is surrounded by $R_2, \ldots, R_7, R_0$ and $n_0 = 3$.

Since the edge between any two components is unique, we use e_{ij} to refer to the edge between R_i and R_j. From Lemma 4.2, we also have that the edges e_{12}, e_{17}, e_{27} can be prolonged inside B and meet at a point. This is impossible since, as e_{10} is vertically straight, e_{12} is going up while e_{17} is going down.
Case 5: R_1 is 8-sided. Hence R_1 is surrounded by $R_2, \ldots, R_8, R_0$. By the same argument as in Case 3 of Lemma 5.3, $p_i > 0$. □

Note that, by Lemma 4.2, we may assume in the previous two proofs that $n_1, \ldots, n_m \geq 4$.

A bubble of type 8m with $p_1 = 0$ tends to have long R_2 and R_7. Hence it does not seem to be minimizing. Furthermore, if p_1 and p_8 are zero, the two external edges of R_2 and of R_7 must have total turning angle $10\pi/3$. This makes both regions a long way away from being chubby as they should be. Consequently, it is unlikely to have a weak minimizer of type 8m with $A(R_1) > A_1$ or $A(R_8) > A_8$.

Proposition 5.7. *Let B be a weakly minimizing 8-bubble for areas $A_1, \ldots, A_8$ with connected regions. Suppose B is not of type 8m with $A(R_1) > A_1$ or $A(R_8) > A_8$. Then B is also minimizing for areas A_i.*

Proof. Suppose B is not of type 8m and 8f. By the previous lemma, $p_i > 0$. By Lemma 3.1, $A(R_i) = A_i$. Suppose B is of type 8m with $A(R_1) = A_1$ and $A(R_8) = A_8$. Since $R_2, \ldots, R_7$ are 4-sided, by Lemma 3.4, $p_2, \ldots, p_7 > 0$. Again, $A(R_i) = A_i$. Suppose B is of type 8f. By Lemma 3.4, $p_2, \ldots, p_8 > 0$ and thus $A(R_i) = A_i$ for $i = 2, \ldots, 8$. Suppose to the contrary that $A(R_1) > A_1$. Hence $p_1 = 0$. Note that every 4-sided component is between zero pressure R_0 and R_1. Recall the definition from [23, 24] that a *circular* component is a 4-sided component with two (opposite) edges on a circle. First, suppose that one of the 4-sided components is not circular. By Proposition 5.24 of [24] ([23, Proposition 5.40]), every 4-sided component is symmetric and hence forms a straight chain, a contradiction. Now suppose that the 4-sided components are circular. By Lemma 5.22 of [24] ([23, Lemma 5.38]), we can move the chain of circular components while fixing length, $A(R_2), \ldots, A(R_8)$ and decreasing $A(R_1)$. If $A(R_1)$ reaches A_1, then the new bubble is a minimizer. Having the same length, B is also a minimizer. If we encounter an illegal meeting, we create a nonminimizing weak enclosure of length $l(B)$, a contradiction. □

Therefore we may conclude that a weak minimizer of type 8f is also minimizing for the same areas. The next proposition will show that a weak minimizer of the excluded type 8m is also minimizing or ties in length to a minimizer for the same areas (satisfying the second condition of Lemma 3.8) provided that there exists a special variation below.

Proposition 5.8. *Let B be a weakly minimizing bubble for areas $A_1, \ldots, A_8$ of type 8m with $A(R_1) > A_1$. Let e be the external edge of R_1 in Figure 7. Suppose that there is a variation $\{B_t | 0 \leq t \leq T\}$ such that $B_0 = B$, B_t is a bubble for $t < T$, areas of $R_2^t, \ldots, R_8^t$ are fixed, the edge e_t is straight and shortening, and B_T has an illegal meeting. Then B has the minimizing length for areas A_i.*

Proof. Consider $t < T$. Since B_t is a bubble and e_t is straight, $p_1^t = 0$. By Lemma 2.3, $\frac{d}{dt} l(B_t) = 0$. Thus $l(B_t) = l(B_0) = l(B)$. First suppose that $A(R_1^t) > A_1$ for $t < T$. Thus, by continuity, $l(B_T) = l(B)$ and $A(R_1^T) \geq A_1$. Hence B_T is a weak enclosure for areas A_i that is not weakly minimizing but with length $l(B)$, a contradiction. Now suppose that $R_1^{t_0}$ has area A_1 at some $t_0 < T$. Hence B_{t_0} is a weak minimizer enclosing areas exactly A_i. Finally B has the same length as the minimizing B_{t_0}. □

By Lemma 3.8, we now obtain the following two main theorems. They show an easier way to prove the planar bubble conjecture for $m = 7$ and 8.

Theorem 5.9. *For $m \leq 7$, if every weakly minimizing bubble has connected regions, then every minimizing bubble has connected regions.*

Proof. This result follows from Propositions 3.2, 5.4 and Lemma 3.8. □

Theorem 5.10. *If every weakly minimizing 8-bubble has connected regions and is not of type 8m with $A(R_1) > A_1$ or $A(R_8) > A_8$, then every minimizing 8-bubble has connected regions.*

Proof. This result follows from Proposition 5.7 and Lemma 3.8. □

For 8-bubbles, we have stronger results (than Proposition 5.7 and Theorem 5.10) with similar proofs as follows.

Proposition 5.11. *Let B be a weakly minimizing 8-bubble for areas with connected regions. Suppose B is not of type 8m with $p_1 = 0$ or $p_8 = 0$. Then B is also minimizing for the same areas.*

Theorem 5.12. *If every weakly minimizing 8-bubble has connected regions and is not of type 8m with $p_1 = 0$ or $p_8 = 0$, then every minimizing 8-bubble has connected regions and is not of type 8m with $p_1 = 0$ or $p_8 = 0$.*

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