SOME NEW CHARACTERISTIC PROPERTIES OF THE A-PEDAL HYPERSURFACES IN E^{n+1}

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Abstract. The primary purpose of this paper is to present the definition of the a-pedal hypersurface with respect to a point in the interior of a closed, convex and smooth hypersurface M. The secondary purpose of this paper is to give some new characteristic properties of the a-pedal hypersurfaces related to the support function, Gauss curvature, mean curvature, the first and second fundamental forms and their coefficients of M (Section 3). Using the classical methods of the hypersurfaces in differential geometry we have established that the support function h_a of the a-pedal hypersurface M_a is equal to $\frac{h^{a+1}}{P_a}$ where $P_a^2 = h^2 + a^2 \overset{III}{\nabla}(h,h)$.

1. Introduction

The notion of the pedal surface of a given surface M in E^3 with respect to a chosen origin is well-known in literature [1, 2, 9, 11]. Georgiou, Hasanis and Koutroufiotis [1] have studied the differential geometry of the pedal surface M with respect to a chosen origin and they investigated the applications in geometrical optics. Recently Kuruoğlu [8] has studied the pedal surface with respect to a point in the interior of a closed, convex and smooth surface M in E^3 and some new characteristic properties of the pedal surface M have been given by the author. Afterwards the pedal surface M in E^3 has been generalized by Kuruoğlu and Sarıoğlugil [9]. Furthermore, some characteristic properties of a-pedal surfaces have been given by Kuruoğlu and Sarıoğlugil [10], the reciprocal surfaces have been studied by Kuruoğlu and Sarıoğlugil in [13], and have been generalized by Sunma, Sarıoğlugil and Kuruoğlu [14].

In this paper, using the method in [13], the a-pedal hypersurfaces with respect to a point in the interior of a closed, convex and smooth surface M are defined and some characteristic properties of reciprocal hypersurface M_a of M are studied.

2. Preliminaries

In this section, we will give a brief review related with the theory of hypersurfaces in E^{n+1} and some characteristic properties and definitions of the hyperpedal and reciprocal hypersurfaces for later use.

Let M be a closed, convex and smooth hypersurface in E^{n+1}. We consider an immersion $\psi\colon M \to E^{n+1}$, pulled back onto the standard metric in E^{n+1}, and make the usual local identifications of M and $\psi(M)$.

We begin by making the following two assumptions [1].

I) The immersed hypersurface in E^{n+1} has Gauss-Kronecker curvature $K = \prod_{i=1}^{n} k_i \neq 0$ everywhere; with k_i denoting the principal curvatures of M.

II) The origin O does not lie on a tangent hyperplane of M. Such an origin will henceforth be called admissible for M. Clearly, admissible origins O always exist locally for a given M. It is sufficient to pick O close enough to M.

If I) and II) hold for $n \geq 2$, there exist an orientation of M, given in the vicinity of any single point by certain ordered n-tuples of coordinates $(u_1, u_2, \ldots, u_n)$, so that the corresponding unit normal vector field

$$N = \frac{X_1 \times X_2 \times \cdots \times X_n}{\|X_1 \times X_2 \times \cdots \times X_n\|} \tag{2.1}$$

points to the half-space which lies in O. Here, $X_i = \frac{\partial X}{\partial u_i}, 1 \leq i \leq n$, and $\times$ is the usual exterior product [1].

The support function h of M with respect to O is defined by

$$h = -\langle X, N \rangle \tag{2.2}$$

where $\langle , \rangle$ is the usual inner product of E^{n+1}.

Assuming we have chosen an admissible origin O, the corresponding support function h clearly never vanishes. Because of connectivity, either $h > 0$ or $h < 0$ is assumed throughout, by assumption II) and the choice of orientation. It follows that we can always choose the unit normal vector field N of M which makes $h > 0$.

Setting $X_i = \frac{\partial X}{\partial u_i}$ and $N_i = \frac{\partial N}{\partial u_i}$ in a chart $(u_1, u_2, \ldots, u_n)$, we can write

$$g_{ij} = \langle X_i, X_j \rangle,\ b_{ij} = -\langle X_i, N_j \rangle = \langle X_{ij}, N \rangle,\ n_{ij} = \langle N_i, N_j \rangle,\ 1 \leq i, j \leq n, \tag{2.3}$$

for the coefficients of the first and second fundamental forms of M, respectively.

Definition 1. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and O be a point in the interior of $P \in M$. If X is the position vector at point $P \in M$ with respect to the origin and is the inner unit vector field of*

M, then the hypersurface with the position vector field with respect to the origin is called hyperpedal surface of and denoted by $\overline{M}$, [9].

Geometrically, we can construct the hyperpedal surface as follows. We draw the perpendicular line from the origin O to the tangent plane $T_M(P)$ and we get the normal to $T_M(P)$. The normal meets $T_M(P)$ at a point $\overline{P}$. The locus of all the points $\overline{P}$ corresponding to all the points P on M will give the hyperpedal surface.

The position vector of the point $\overline{P} \in \overline{M}$ can be given by

$$\overline{X} = -hN. \tag{2.4}$$

Thus for the position vector field X of M we can write

$$X = X_T + X_N \tag{2.5}$$

where X_T and $X_N = \overline{X} = -hN$ denote the decompositions of tangential and normal of X, respectively.

Furthermore, because M is strictly convex, we can express it locally in terms of the inverse tensor (n^{ik}) of the third fundamental form $III = (n_{ik})$ of M, with respect to arbitrary parameter system, namely

$$X = -hN - \sum_{i,k} n^{ik} h_i N_k, \ 1 \leq i, k \leq n, \tag{2.6}$$

where h_i, N_k are the partial derivatives with respect to the local parameters [9]. The shape operator S is the self-adjoint linear transformation defined by $S(V) = D_V N$ for all $V \in T_M(P)$. Using equation (2.5), we can write

$$\text{grad } \rho = \frac{1}{\rho} X \text{ and grad } h = SX_T \tag{2.7}$$

where $X_T \in T_M(P)$ and $\|X\| = \rho$ [2].

Furthermore, the qth-order Gauss curvature K_q of M is defined by

$$K_q = \sum_{i_1 \leq i_2 \leq \cdots \leq i_q}^{n} k_{i_1} k_{i_2} \cdots k_{i_q}, \quad 1 \leq q \leq n, \tag{2.8}$$

where $k_1, k_2, \ldots, k_n$ are the eigenvalues of S [4].

Here, K_1 and K_n are the mean and Gauss curvatures of M and denoted by H and K. If the Gauss curvature K of M is constant, we can say that M is a hypersurface with constant curvature.

Furthermore, the volume V of M may be written as

$$V = \frac{1}{n+1} \int_M h dA \tag{2.9}$$

where h is the support function of M [11].

Lemma 1. *Let M be a hypersurface of E^{n+1}. We list the following relations between the higher order Gauss curvature functions and the principal curvature functions of M.*

$$\begin{aligned} K_p^{(p)} + k_{p+1} &= K_{p+1}^{(p+1)} \\ k_{p+1} + K_1^{(p)} &= K_1^{(p+1)}, \quad 1 \leq p+1 \leq n \\ k_{p+1} K_{r-1}^{p} + K_r^{(p)} &= K_r^{(p+1)}. \end{aligned} \tag{2.10}$$

Here $K_r^{(p)}$ and $K_r^{(p+1)}$ is not defined on the same manifold. For example, $K_r^{(p)}$ and $K_r^{(p+1)}$ are defined on p-dimensional and $(p+1)$-dimensional manifolds, respectively. We know that p-dimensional manifold is included in the $(p+1)$-dimensional manifold [5].

Definition 2. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and O be a point in the interior of M. If X is the position vector at point $P \in M$ with respect to the origin O and N is the inner unit vector field of M, then the hypersurface with the position vector field $X_{rc} = -\frac{1}{h}N$ with respect to the origin O is called reciprocal hypersurface of M and denoted by M_{rc} [14].*

3. Some New Characteristic Properties of the a-pedal Hpersurfaces in E^{n+1}

In this section we will give the definition and some new characteristic properties of the a-pedal hypersurfaces in E^{n+1}.

Definition 3. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and O be a point in the interior of M. The a-pedal of M is the hypersurface having the position vector field*

$$X_a = -h^a N \tag{3.1}$$

with respect to the origin O. Here, N is the unit normal vector field of M at the point $P \in M$ and h is the support function of M.

Theorem 1. *Let M be a closed, convex and smooth hypersurface in E^{n+1}. For the unit normal vector field N_a of the a-pedal hypersurface M_a we have*

$$N_a = \frac{1}{P_a}\{(a+1)hN + aX\} \tag{3.2}$$

where X is the position vector field of M and $P_a^2 = h^2 + a^2 \overset{III}{\nabla}(h,h)$.

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be a local coordinate system on M. By differentiating the position vector field X_a with respect to the parameter u_i, $1 \leq i \leq n$, we get

$$(X_a)_i - ah^{a-1}h_i N - h^a N_i, \quad 1 \leq i \leq n. \tag{3.3}$$

Thus, using (2.1) for the unit normal vector field N_a we can write

$$N_a = \frac{(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n}{\|(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n\|}. \tag{3.4}$$

By computing the vector field $(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n$ we obtain

$$\begin{aligned}&(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n \\ &\quad = h^{na-1} \|X_1 \times X_2 \times \cdots \times X_n\| \left(hN - a \sum_{i=1}^{n} n \frac{h_i}{k_i g_{ii}} X_i \right)\end{aligned}$$

and using (2.6) we can rewrite the vector field $(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n$ as

$$(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n = h^{na-1} \|X_1 \times X_2 \times \cdots \times X_n\| ((a+1)hN + aX). \tag{3.5}$$

On the other hand, by computing the norm of the vector field we get

$$\|(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n\| = h^{na-1} \|X_1 \times X_2 \times \cdots \times X_n\| P_a. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we get the result of the theorem. □

Theorem 2. *Let M be a closed, convex and smooth hypersurface in E^{n+1}. For the unit normal vector field h_a of the a-pedal hypersurface M_a we have*

$$h_a = \frac{h^{a+1}}{P_a}. \tag{3.7}$$

Proof. Using the definition of the support function, (3.1) and (3.2), the proof of this theorem can be easily shown. □

Theorem 3. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and M_a be the a-pedal hypersurface of M. For the hyper-area element dA_a of M_a we have*

$$dA_a = h^{na-1} K P_a dA \tag{3.8}$$

where K is the Gauss curvature of M.

Proof. For the hyper-area element dA_a of M_a

$$dA_a = \|(X_a)_1 \times (X_a)_2 \times \cdots \times (X_a)_n\| du_1 du_2 \cdots du_n \tag{3.9}$$

where $\{u_i : 1 \leq i \leq n\}$ is a local coordinate system on M. Substituting (3.6) into (3.9) we get

$$dA_a = h^{na-1} K P_a \|X_1 \times X_2 \times \cdots \times X_n\| du_1 du_2 \ldots du_n.$$

Setting $dA = \|X_1 \times X_2 \times \cdots \times X_n\| du_1 du_2 \ldots du_n$ in the equation above, we obtain the result of the theorem. □

Theorem 4. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and M_a be the a-pedal hypersurface of M.*

For the coefficient $(g_a)_{ij}$, $1 \leq i,j \leq n$ of the first fundamental form I_a of M_a we can write

$$(g_a)_{ij} = h^{2(a-1)} \left\{ a^2 h_i h_j + h^2 n_{ij} \right\} \tag{3.10}$$

where n_{ij} is the coefficient of the third fundamental form III of M.

Proof. Using (2.3), for the coefficient $(g_a)_{ij}$ of the first fundamental form I_a of M_a we can write

$$(g_a)_{ij} = \langle (X_a)_i, (X_a)_J \rangle, 1 \leq i,j \leq n. \tag{3.11}$$

Then, by differentiating the position vector field with respect to the parameter we obtain

$$(X_a)_j = -ah^{a-1} h_j N - h^a N_j,\ 1 \leq j \leq n. \tag{3.12}$$

Substituting (3.3) and (3.12) into (3.11) we get

$$(g_a)_{ij} = h^{2(a-1)} \left\{ a^2 h_i h_j + h^2 n_{ij} \right\}\ , 1 \leq i,j \leq n.$$

This completes the proof. □

Thus, we can give the following lemma.

Lemma 2. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and M_a be the a-pedal hypersurface of M. Then we have*

$$\det (g_a)_{ij} = h^{na-1} K P_a \det\ g_{ij}. \tag{3.13}$$

Theorem 5. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and M_a be the a-pedal hypersurface of M. For the $(b_a)_{ij}$ coefficient of the second fundamental form II_a of M_a we can write*

$$(b_a)_{ij} \ = \frac{h^{a-1}}{P_a} \left\{ a(a+1) h_i h_j - ahb_{ij} + (a+1) h^2 n_{ij} \right\} \tag{3.14}$$

where $b_{i,j}$ is the coefficient of the second fundamental form II of M.

Proof. Using (2.3) for the $(b_a)_{ij}$ coefficient of the second fundamental form II_a of M_a we can write

$$(b_a)_{ij} \ = \langle (X_a)_{ij}, N \rangle,\ 1 \leq i,j \leq n. \tag{3.15}$$

By differentiating the position vector field $(X_a)_i$ with respect to the parameter u_j we have

$$(X_a)_{ij} = -a[(a-1) h_i h_j + ah^{a-1} b_{ij}] - ah^{a-1} h_i N_j - ah^{a-1} h_j N_i - h^a N_{ij}. \tag{3.16}$$

Substituting (3.2) and (3.16) into (3.15) and by rearranging the last equation obtained, we get the result of the theorem. □

Theorem 6. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and M_a be the a-pedal hypersurface of M. For the principal curvatures $(k_a)_i,\ 1 \leq i \leq n-1$ and $(k_a)_n$ of M_a we have*

$$(k_a)_i = \frac{1}{h^a P_a}\left[(a+1)h - \frac{a}{k_i P_a}\right] \tag{3.17}$$

and

$$(k_a)_n = \frac{1}{ah^{a-1}P_a}\left[(a+1)(1+\frac{a-1}{P_a^2}h^2) - \frac{h}{k_n P_a^2}\right] \tag{3.18}$$

where k_i, $1 \leq i \leq n$, is the ith principal curvature of M.

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be a local parameter system consisting of the curvature lines on M. Since $Y = \sum_{i=1}^n \lambda_i \frac{\partial}{\partial u_i} \in \chi(M)$ we have

$$(dX_a)(Y) = \sum_{i=1}^n \lambda_i (X_a)_i \tag{3.19}$$

where X_a is the position vector field of M_a. Substituting (3.4) into (3.19), we find

$$(dX_a)(y) = -ah^{a-1}\left[\sum_{i=1}^n \lambda_i h_i\right] N - h^a \sum_{i=1}^n \lambda_i N_i. \tag{3.20}$$

Substituting $\sum_{i=1}^n \lambda_i h_i = \langle Y, \text{grad } h\rangle$ and $N_i = -k_i X_i$ into (3.20), we obtain

$$(dX_a)(Y) = -ah^{a-1}\langle Y, \text{grad } h\rangle N + h^a SY$$

where S is the shape operator of M.

Setting grad $h = St$ in the equation above, we get

$$(dX_a)(Y) = -ah^{a-1}\langle Y, St\rangle\ N + h^a SY. \tag{3.21}$$

Then for the unit normal vector field N_a we can write

$$(dN_a)(Y) = \sum_{i=1}^n \lambda_i (N_a)_i. \tag{3.22}$$

By differentiating the vector field N_a with respect to the parameter u_i we get

$$(N_a)_i = \frac{(a+1)(h_i N + hN_i) + aX_i}{P_a} + \frac{(a+1)hN + aX}{P_a^2}(P_a)_i,\ 1 \leq i \leq n. \tag{3.23}$$

By substituting (3.23) into (3.22) and by rearranging terms of the last equation obtained, we get

$$(dN_a)(Y) = \frac{1}{P_a}\left[\sum_{i=1}^{n}\lambda_i\left[(a+1)(h_iN+hN_i)+aX_i\right]\right]N$$
$$-\frac{a+1}{P_a^2}h\left[\sum_{i=1}^{n}\lambda_i(P_a)_i\right]N-\frac{a}{P_a^2}\left[\sum_{i=1}^{n}\lambda_i(P_a)_i\right]X. \tag{3.24}$$

Setting $\sum_{i=1}^{n}\lambda_i(P_a)_i = \langle Y,\text{grad } P_a\rangle$ and $\sum_{i=1}^{n}\lambda_i h_i = \langle Y,\text{grad } h\rangle$ into (3.24), we get

$$(dN_a)(Y) = \frac{a+1}{P_a}\langle Y,\text{grad } h\rangle N - \frac{a+1}{P_a^2}h\langle Y,\text{grad } P_a\rangle N$$
$$-\frac{a}{P_a^2}\langle Y,\text{grad } P_a\rangle X - \frac{a+1}{P_a}hSY + \frac{a}{P_a}Y.$$

By writing $X = t - hN$ in the equation above, we obtain

$$(dN_a)(Y) = \frac{a+1}{P_a}\langle St,Y\rangle N - \frac{1}{P_a^2}h\langle Y,\text{grad } P_a\rangle N$$
$$-\frac{a}{P_a^2}\langle Y,\text{grad } P_a\rangle t - \frac{a+1}{P_a}hSY + \frac{a}{P_a}Y. \tag{3.25}$$

Setting $\langle Y,\text{grad } P_a\rangle = \frac{1-a^2}{P_a^2}h\langle St,Y\rangle + \frac{a^2tY}{P_a}$ in (3.25) we get

$$(dN_a)(Y) = \left\{\frac{a+1}{P_a}\langle St,Y\rangle - \frac{a^2-1}{P_a^3}h^2\langle St,Y\rangle - \frac{ha^2}{P_a^3}\langle Y,t\rangle\right\}N$$
$$-\frac{a+1}{P_a}hSY + \frac{a}{P_a}Y + \frac{a}{P_a^3}\left\{(a^2-1)h\langle St,Y\rangle - a^2\langle Y,t\rangle\right\}t.$$

By the Olinde-Rodriques formula we may write

$$(dN_a)(Y) + k_a(Y)(dX_a)(Y) = 0. \tag{3.26}$$

By substituting (3.21) and (3.25) into (3.26), we obtain

$$\left[\left\{\frac{a+1}{P_a}\langle St,Y\rangle - \frac{a^2-1}{P_a^3}h^2\langle St,Y\rangle - \frac{ha}{P_a^3}\langle Y,t\rangle\right\}N\right.$$
$$\left.+\frac{a}{P_a^3}\left\{(a^2-1)h\langle St,Y\rangle - a^2\langle Y,t\rangle\right\}t - \frac{a+1}{P_a}hSY + \frac{a}{P_a}Y\right]$$
$$+ \quad k_a(Y)\left[-ah^{a-1}\langle Y,St\rangle\ N + h^aSY\right] = 0.$$

From the equation above, the following linear equation system becomes

$$a(k_a)h^{a-1}\langle St,Y\rangle = \frac{a+1}{P_a}\langle St,Y\rangle - \frac{a^2-1}{P_a^3}h^2\langle St,Y\rangle - \frac{ha^2}{P_a^3}\langle Y,t\rangle$$

and

$$(k_a)h^a SY = \frac{a+1}{P_a} hSY - \frac{a}{P_a} Y - \frac{a}{P_a^3} \left\{ (a^2-1)h\langle St, Y\rangle - a^2 \langle Y, t\rangle \right\} t.$$

Setting $Y = X_i$ and $SX_n = k_n X_n$ in the first equation of the linear equation system above we get

$$(k_a)_n = \frac{1}{ah^{a-1}P_a} \left[(a+1)(1 + \frac{a-1}{P_a^2} h^2) - \frac{h}{k_n P_a^2} \right].$$

Setting $Y = X_i,\ 1 \leq i \leq n-1$, and $\langle t, X_i = 0 \rangle$ in the second equation of the linear equation system above we get

$$(k_a)_i = \frac{1}{h^a P_a} \left[(a+1)h - \frac{a}{k_i P_a} \right].$$

This completes the proof. □

Theorem 7. *Let M be a closed, convex and smooth hypersurface in E^{n+1} and M_a be the a-pedal hypersurface of M. For the qth order Gauss curvature $(K_a)_q^n$ of M_a we have*

$$\begin{aligned}(K_a)_q^n = \frac{1}{h^{qa} P_a^q K_n^{(n)}} &\binom{n-1}{q-1} \left[\frac{n-q}{q} \lambda + \frac{h}{a} \eta - \frac{h^2}{a P_a^2 k_n} \right] K_n^{(n)} \lambda^{q-1} \\ &+ (-1)^q \, a^q k_n K_{n-q-1}^{(n-1)} \left(\frac{h^2}{P_a^2} - (h\eta - a\lambda) k_n \right) \\ &\times \sum_{i=1}^{q-1} (-1)^{i+1} \binom{q-1}{i} a^{i-1} \lambda^{q-i-1} K_{n-i-1}^{(n-1)} \end{aligned} \qquad (3.27)$$

where $K_{n-q-1}^{(n-1)}$ and $K_{(n-1)-(q-1)}^{(n-1)}$ are the $[(n-1)-q]$th and $[(n-1)-(q-1)]$th Gauss curvatures of M, respectively. Here, $\lambda = (a+1)h$ and $\eta = (a+1)(1+\frac{a-1}{P_a^2}h^2)$.

Proof. Using (2.8), for the qth order Gauss curvature $(K_a)_q^n$ of M_a we may write

$$(K_a)_q^n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_q}^{n} (k_a)_{i_1} (k_a)_{i_2} \cdots (k_a)_{i_q}, \ 1 \leq q \leq n.$$

By expanding the equation above we obtain

$$\begin{aligned}(K_a)_q^n = &\sum_{i_1 \leq i_2 \leq \cdots \leq i_q}^{n-1} (k_a)_{i_1} (k_a)_{i_2} \cdots (k_a)_{i_q} \\ &+ (k_a)_n \sum_{i_1 \leq i_2 \leq \cdots \leq i_{q-1}}^{n-1} (k_a)_{i_1} (k_a)_{i_2} \cdots (k_a)_{i_{q-1}}. \end{aligned} \qquad (3.28)$$

By substituting (3.17) and (3.18) into (3.28) and rearranging the last equation obtained we get the result of the theorem. □

Theorem 8. *Let M be a closed, convex and smooth hypersurfaces in E^{n+1} and M_a be the a-pedal hypersurfaces of M. For the volume V_a of M_a, we have*

$$V_a = \frac{1}{n+1}\int_M h^{a(n+1)} K dA \tag{3.29}$$

where K is the Gauss curvature of M.

Proof. Using (2.9), the volume V_a of M_a, we can write

$$V_a = \frac{1}{n+1}\int_M h_a dA_a.$$

By substituting (3.8) and (3.9) into the equation above, we get the result of theorem. □

Remark 1. *For $a = -1$, the a-pedal hypersurface is a reciprocal hypersurface. Setting $a = -1$ in the equations above we obtain the results in [14].*

Similarly, for $a = 1$ the a-pedal hypersurface is a pedal hypersurface. Thus, we get the results in [9].

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