

ISOPERIMETRIC REGIONS IN THE HYPERBOLIC PLANE BETWEEN PARALLEL HOROCYCLES

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ABSTRACT. In this work the following problem in the hyperbolic plane is investigated. Find the perimeter-minimizing regions of prescribed area between two parallel horocycles. An explicit and detailed description of all such regions is given through isoperimetric inequalities.

1. INTRODUCTION

For a Riemannian manifold M , the classical isoperimetric problem consists in classifying, up to congruency by the isometry group of M , the (compact) regions $\Omega \subseteq M$ enclosing a fixed volume that have minimal boundary volume. The existence and regularity of solutions for a large number of cases may be guaranteed by adapting some results from Geometric Measure Theory (cf. [9]).

When M is the Euclidean plane \mathbb{R}^2 , the classical isoperimetric problem has the disk as the unique solution. After the simply connected spaces of constant curvature, slabs are the most natural ambient to work in. If M is a hyperbolic surface, the least-perimeter enclosures of prescribed area are described in [1], [7] and [11]. Physically, the isoperimetric problem in a slab corresponds to determining the shape of a drop trapped between two parallel planes, which was solved by Vogel in [12]. Independently, Athanassenas studied the isoperimetric problem between parallel planes of \mathbb{R}^3 in [2]. If M is a slab between two parallel horospheres in the 3-dimensional hyperbolic space $\mathbb{H}^3(-1)$, the possible isoperimetric regions were obtained in [3].

In this paper, the upper halfplane model \mathbb{R}_+^2 will be used to investigate the isoperimetric problem when M is a slab between two parallel horocycles of \mathbb{R}_+^2 , represented by horizontal straight lines. A detailed and complete classification of the isoperimetric solutions will be presented. The main result of the paper is part of the doctoral thesis [5] presented at University of São Paulo/Brazil in September, 2006, and is available in Portuguese from the CAPES database and posted at arXiv.org in 2009. It has been

generalized to regions between any constant-curvature curves in \mathbb{R}^2 , \mathbb{S}^2 , or \mathbb{H}^2 in subsequent independent work by M. Simonson [11] using another approach. Furthermore, this paper is focused on the case when the two boundary curves have the same curvature.

In Section 2 some basic definitions for \mathbb{R}_+^2 are given in order to get some preliminary characterizations of the possible isoperimetric regions. More explicitly they must be delimited by curves of constant geodesic curvature and meet the horocycles perpendicularly when this intersection is non-empty. Although this result is partially adapted from [3], the techniques used in \mathbb{R}_+^2 to determine the isoperimetric regions are very different.

Section 3 contains important results proved in [4] for the possible isoperimetric regions obtained in the previous section.

Section 4 is a fundamental part of the paper because the inequalities which will be used in Section 5 are listed there. In this last section the isoperimetric profile for the region between two parallel horocycles in \mathbb{R}_+^2 is studied to prove the following *sharp* result.

Let $c > 1$ be a real constant and $\mathcal{F}_c = \{(x, y) \in \mathbb{R}_+^2 : 1 \leq y \leq c\}$. Let $A > 0$ and $\mathcal{C}_{c,A}$ be the set of all $\Omega \subset \mathcal{F}_c$ with area $|\Omega| = A$ and perimeter $|\partial(\Omega \cap \overset{\circ}{\mathcal{F}}_c)| < \infty$, where Ω is supposed to be connected, compact and 2-rectifiable in \mathcal{F}_c , having as boundary (between the horocycles) a simple rectifiable curve.

Theorem 1.1. *Let $L_{c,A} = \inf\{|\partial(\Omega \cap \overset{\circ}{\mathcal{F}}_c)| : \Omega \in \mathcal{C}_{c,A}\}$. Then*

- (1) *there exists $\Omega \in \mathcal{C}_{c,A}$ such that $|\partial(\Omega \cap \overset{\circ}{\mathcal{F}}_c)| = L_{c,A}$;*
- (2) *if $\Omega \subset \mathcal{F}_c$ has minimal perimeter, the boundary of Ω has a single connected component consisting of either*
 - (a) *a halfdisk (geodesic, horocycle, equidistant) above $\{y = 1\}$;*
 - (b) *a section of \mathcal{F}_c , namely*

$$S_{[x_0, x_1]} = [x_0, x_1] \times [1, c].$$

More precisely, if d is the hyperbolic distance between the horocycles then

- i. *if $d < 1$, there exists $A_0(c)$ such that*
 - *if $A < A_0(c)$ then Ω is a geodesic halfdisk;*
 - *if $A = A_0(c)$ then Ω is a geodesic halfdisk or a section;*
 - *if $A > A_0(c)$ then Ω is a section;*
- ii. *if $d = 1$, there exists $A_0(c)$ such that*
 - *if $A < A_0(c)$ then Ω is a geodesic halfdisk;*
 - *if $A = A_0(c)$ then Ω is a horocycle halfdisk or a section;*
 - *if $A > A_0(c)$ then Ω is a section;*
- iii. *if $d > 1$, there exist two constants $A_0(c) < A_1(c)$ such that*
 - *if $A < A_0(c)$ then Ω is a geodesic halfdisk;*
 - *if $A = A_0(c)$ then Ω is a horocycle halfdisk;*

- if $A_0(c) < A < A_1(c)$ then Ω is an equidistant halfdisk;
- if $A = A_1(c)$ then Ω is an equidistant halfdisk or a section;
- if $A > A_1(c)$ then Ω is a section.

2. PRELIMINARIES

This section contains some basic facts and notations that will be used throughout the paper. There is much literature about the subject (for instance, [6]).

Let $\mathcal{L}^3 = (\mathbb{R}^3, g)$ be the 3-dimensional Lorentz space endowed with the metric $g(x, y) = x_1y_1 + x_2y_2 - x_3y_3$ and the hyperbolic plane

$$\mathbb{H}^2 := \{p = (x_1, x_2, x_3) \in \mathcal{L}^3 : g(p, p) = -1, x_3 > 0\}.$$

The upper halfplane model $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2; y > 0\}$ for \mathbb{H}^2 will be used, endowed with the metric

$$\langle, \rangle = ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The Euclidean straight line $\{y = 0\}$ is the infinity boundary of \mathbb{R}_+^2 , denoted by $\partial_\infty \mathbb{R}_+^2$.

The curves of constant geodesic curvature $k \geq 0$ in \mathbb{R}_+^2 are classified as *geodesic*, *geodesic circles*, *horocycles* and *equidistant curves*, which are classical terms detailed, for instance, in [4] and [6].

Let \mathcal{F} be the region inside two parallel horocycles (represented by two horizontal Euclidean straight lines). Since the Euclidean homothety is an isometry of \mathbb{R}_+^2 , the lower horocycle can be chosen as $\{y = 1\}$ to study the isoperimetric problem, so that any solution is obtained by homothety. Recall that \mathcal{F}_c was defined in the Introduction. From now on \mathcal{F} will be denoted by \mathcal{F}_c . The isoperimetric problem for \mathcal{F}_c may be formulated as follows.

Fix an area value and study the domains $\Omega \subset \mathcal{F}_c$ with the prescribed area which have minimal free boundary perimeter, but not counting its part of the boundary contained in the horocycles.

Definition 2.1. *A (compact) minimizing region Ω for this problem will be called an isoperimetric solution or region in \mathcal{F}_c .*

Remark 2.1. *By adapting the demonstration of Theorem 1.1 from [3] to the case studied in this paper, namely \mathbb{R}_+^2 , together with Lemma 2.1 of [1], there exist regular isoperimetric solutions and they are regions whose boundary consists of curves of constant geodesic curvature perpendicular to the horocycles (when the intersection is non-empty). Essentially, this proves the first item of Theorem 1.1 in this present paper, stated in the Introduction.*

Therefore there are only the following possibilities for barriers: vertical geodesics, geodesic circles, horocycles represented by Euclidean circles of \mathbb{R}_+^2 tangent to $\partial_\infty \mathbb{R}_+^2$, and equidistant curves represented by Euclidean circles not entirely contained in \mathbb{R}_+^2 and neither tangent nor perpendicular to $\{y = 0\}$. A region in \mathcal{F}_c delimited by two vertical geodesics will be called a *section*. A region in \mathcal{F}_c delimited by geodesic circles perpendicular to $\{y = 1\}$ or $\{y = c\}$ will be called a *geodesic halfdisk*. A region in \mathcal{F}_c delimited by horocycles and equidistant curves perpendicular to $\{y = 1\}$ will be called a *horocycle halfdisk* and an *equidistant halfdisk*, respectively. *Halfdisk above* (respectively *below*) $\{y = c\}$ means the part of the Euclidean halfdisk above (respectively below) the horocycle $\{y = c\}$.

3. EXPRESSIONS FOR PERIMETER AND AREA

This section contains expressions for the perimeter and area of the possible isoperimetric solutions Ω in \mathcal{F}_c . All results of this section are proved in detail in [4] but brief sketches of the proofs are given here. For the purposes of this paper only regions that are 2(-dimensional)-rectifiable (with respect to Hausdorff measure) with boundary 1(-dimensional)-rectifiable will be considered. This measure is denoted by $|\cdot|$, so that any Ω has area $|\Omega|$ and perimeter $|\partial\Omega|$, but it *never* counts $\partial\Omega \cap \partial\mathcal{F}_c$. See [9] for more details.

3.1. Perimeter and area of a section. Let $c > 1$ and $x_0 < x_1$ be real constants. For the sake of simplicity, consider the vertical geodesics $\{x = x_0\}$ and $\{x = x_1\}$ contained in \mathbb{R}_+^2 , and the parallel horocycles $\{y = 1\}$ and $\{y = c\}$.

Lemma 3.1. *Using the notation above, if T is a section then*

$$|\partial T| = 2 \ln c \quad \text{and} \quad |T| = (x_1 - x_0)(-1/c + 1).$$

Proof. Since the length of a vertical geodesic segment $1 < y < c$ is $\ln(c/1) = \ln c$, then $|\partial T| = 2 \ln c$ and

$$|T| = \int_{x_0}^{x_1} \int_1^c \frac{1}{y^2} dy dx = (x_1 - x_0)(-1/c + 1).$$

□

3.2. Perimeter and area for a geodesic halfdisk and a horocycle halfdisk. Now suppose $c > 1$ and consider the parallel horocycles $\{y = 1\}$ and $\{y = c\}$. Let S_1 be the circle centered at $(0, 1)$ with radius $r_1 < 1$, and S_2 the circle centered at $(0, c)$ with radius $r_2 < c - 1$ (see Figure 1). Hence, S_1 can be viewed as a geodesic circle S_H^1 with hyperbolic center $C_H^1 = (0, h_1)$ and S_2 as a geodesic circle S_H^2 with hyperbolic center $C_H^2 = (0, h_2)$.

Let β_1 be the central angle of S_H^1 corresponding to the arc above $\{y = 1\}$ and β_2 the central angle of S_H^2 corresponding to the arc below $\{y = c\}$.

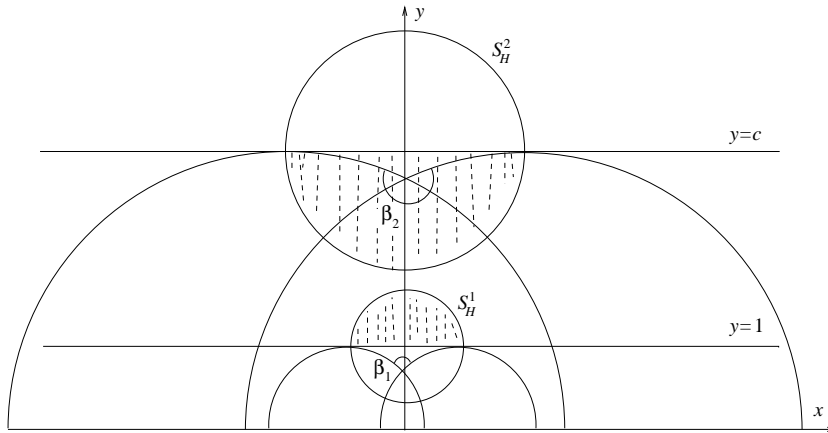


FIGURE 1. Perimeter and area for geodesic halfdisks.

For geodesic halfdisks the following result holds (see Figure 1):

Lemma 3.2. *Using the notation above, let \tilde{S}_1 be the geodesic through C_H^1 and $(r_1, 1)$, and \tilde{S}_2 the geodesic through C_H^2 and (r_2, c) . Let $\theta_1 = \beta_1/2$ and $\theta_2 = \pi - \beta_2/2$ with $0 < \theta_1, \theta_2 < \pi/2$. Let S_1^+ be the geodesic halfdisk delimited by S_H^1 and above $\{y = 1\}$, and S_2^- the halfdisk delimited by S_H^2 and below $\{y = c\}$. Then*

$$|\partial S_1^+| = 2\theta_1 \cot \theta_1, \quad |\partial S_2^-| = 2(\pi - \theta_2) \cot \theta_2, \quad (1)$$

and

$$|S_1^+| = \frac{2\theta_1}{\sin \theta_1} - \pi + 2 \cos \theta_1, \quad |S_2^-| = \frac{2(\pi - \theta_2)}{\sin \theta_2} - \pi - 2 \cos \theta_2. \quad (2)$$

Proof. The arclengths determined by β_1 and β_2 are

$$|\partial S_1^+| = \beta_1 \sinh \rho_1, \quad |\partial S_2^-| = \beta_2 \sinh \rho_2.$$

But

$$\sinh \rho_1 = \frac{r_1}{\sqrt{1 - r_1^2}} = \cot \theta_1, \quad \sinh \rho_2 = \frac{r_2}{\sqrt{c^2 - r_2^2}} = \cot \theta_2,$$

so that the first part of the lemma is proved.

Now observe that

$$|S_1^+|/2 = |\tilde{S}_1| - |\bar{S}_1|,$$

where \tilde{S}_1 is the sector corresponding to θ_1 and \bar{S}_1 is the region delimited by \tilde{S}_1 , axis y and the horocycle $\{y = 1\}$. In the same way,

$$|S_2^-|/2 = |\tilde{S}_2| + |\bar{S}_2|,$$

where \tilde{S}_2 is the sector corresponding to $\beta_2/2 = \pi - \theta_2$ and \bar{S}_2 is the region delimited by \tilde{S}_2 , axis y and the horocycle $\{y = c\}$.

Therefore, the area of S_1^+ and S_2^- are given by

$$\begin{aligned} |S_1^+| &= 2\theta_1 (\cosh \rho_1 - 1) - 2(-r_1 + \pi/2 - \arcsin(h_1)), \\ |S_2^-| &= 2(\pi - \theta_2) (\cosh \rho_2 - 1) + 2(-r_2/c + \pi/2 - \arcsin(h_2/c)). \end{aligned} \tag{3}$$

But

$$\cosh \rho_1 = \frac{1}{\sqrt{1 - r_1^2}}, \quad \cosh \rho_2 = \frac{c}{\sqrt{c^2 - r_2^2}}. \tag{4}$$

Furthermore,

$$\begin{aligned} \arccos(r_1) &= \arcsin(\sqrt{1 - r_1^2}) = \arcsin(h_1), \\ \arccos(r_2/c) &= \arcsin\left(\frac{\sqrt{c^2 - r_2^2}}{c}\right) = \arcsin(h_2/c). \end{aligned} \tag{5}$$

Since $\cos \theta_1 = r_1$ and $\cos \theta_2 = r_2/c$

$$\sin \theta_1 = \sqrt{1 - r_1^2}, \quad \sin \theta_2 = \frac{\sqrt{c^2 - r_2^2}}{c}. \tag{6}$$

By (3), (4), (5), and (6) the proof of (2) is complete. □

Corollary 3.1. *Let H be the horocycle halfdisk above $\{y = 1\}$ represented by a Euclidean semicircle with center $(0, 1)$ and radius 1. Then*

$$|\partial H| = 2 \quad \text{and} \quad |H| = 4 - \pi. \tag{7}$$

Proof. It is enough to calculate $|\partial S_1^+|$ and $|S_1^+|$ from (1) and (2) for the limiting case when $\theta_1 \rightarrow 0$. □

3.3. Perimeter and area for an equidistant halfdisk. Let \bar{E} be the equidistant curve represented by a Euclidean circle with center $(0, 1)$ and radius $r > 1$. The Euclidean equation of \bar{E} is given by $x^2 + (y - 1)^2 = r^2$. Then $\bar{E} \cap \partial_\infty \mathbb{R}_+^2 = \{(-\sqrt{r^2 - 1}, 0), (\sqrt{r^2 - 1}, 0)\}$ (see Figure 2). The curve \bar{E} is equidistant from the geodesic η with equation $x^2 + y^2 = r^2 - 1$. If ρ denotes the hyperbolic distance between \bar{E} and η , then

$$r = \coth \rho. \tag{8}$$

If α is the non-oriented angle between \bar{E} and η , $0 < \alpha < \pi/2$, then (for instance, see Proposition 3 in Chapter 5 of [6])

$$\tanh \rho = \sin \alpha. \tag{9}$$

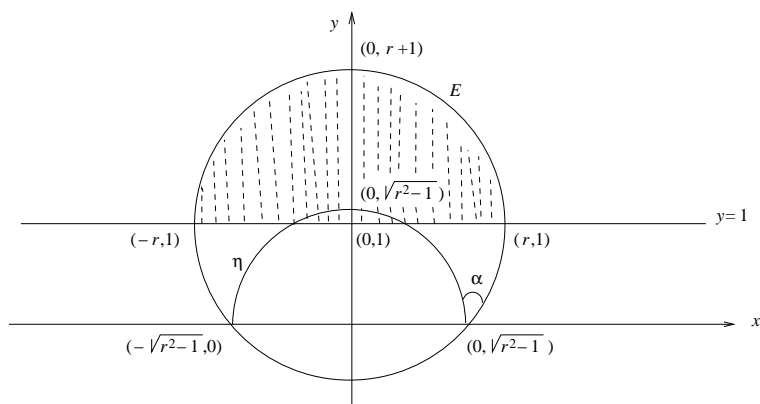


FIGURE 2. Perimeter and area for an equidistant halfdisk.

Lemma 3.3. *Using the notation above, let E be the equidistant halfdisk above $\{y = 1\}$. Then*

$$|\partial E| = \frac{2}{\cos \alpha} \ln \left(\frac{1}{\sin \alpha} + \cot \alpha \right), \tag{10}$$

$$|E| = \frac{2}{\sin \alpha} - \pi + \frac{2}{\cot \alpha} \ln \left(\frac{1}{\sin \alpha} + \cot \alpha \right).$$

Proof. In order to calculate $|\partial E|$, E can be parametrized by

$$\beta(t) = (r \cos t, 1 + r \sin t), \quad 0 \leq t \leq \pi.$$

Then

$$|\partial E| = 2 \int_0^{\pi/2} \frac{r}{1 + r \sin t} dt = \frac{2r}{\sqrt{r^2 - 1}} \ln(r + \sqrt{r^2 - 1}).$$

By (8) and (9), it holds that $r = 1/\sin \alpha$, hence $\sqrt{r^2 - 1} = \cot \alpha$, because $0 < \alpha < \pi/2$. Therefore,

$$|\partial E| = \frac{2}{\cos \alpha} \ln \left(\frac{1}{\sin \alpha} + \cot \alpha \right),$$

and the first part of (10) is proved. Now,

$$|E| = 2 \int_0^r \int_1^{1+\sqrt{r^2-x^2}} \frac{1}{y^2} dy dx = 2r - \pi + \frac{1}{\sqrt{r^2-1}} \ln |(r + \sqrt{r^2-1})^2|.$$

By (8) and (9), it follows that $|E|$, as function of the equidistance angle α , is given by

$$|E| = \frac{2}{\sin \alpha} - \pi + \frac{2}{\cot \alpha} \ln \left(\frac{1}{\sin \alpha} + \cot \alpha \right),$$

which proves the second part of (10). \square

4. COMPARISON OF PERIMETERS OF REGIONS WITH PRESCRIBED AREA

In this section the perimeter and the area of regions delimited by curves of constant geodesic curvature are compared. Their isoperimetric profiles in \mathcal{F}_c will be obtained in the next section as functions of its hyperbolic width d . Since we have been considering the horocycles $\{y = 1\}$ and $\{y = c\}$, the constant c must satisfy the following condition: if H is a horocycle halfdisk above $\{y = c\}$ and T is a section in \mathcal{F}_c , then $|\partial H| = |\partial T|$. By (7) and Lemma 3.1, this means $2 = 2 \ln c$, hence $c = e$ and $d = 1$. This is why d is compared with 1 in Theorem 1.1.

From the geometric analysis done in [4] for the perimeter and area of the possible isoperimetric solutions, there are only the following cases to consider:

- (1) to compare a geodesic halfdisk above $\{y = 1\}$ with a geodesic disk entirely contained in \mathcal{F}_c ;
- (2) to compare a geodesic halfdisk above $\{y = 1\}$ with a geodesic halfdisk below $\{y = c\}$;
- (3) to compare a horocycle halfdisk above $\{y = 1\}$ with a geodesic halfdisk below $\{y = c\}$;
- (4) to compare an equidistant halfdisk above $\{y = 1\}$ with a geodesic halfdisk below $\{y = c\}$.

In order to prove the second part of Theorem 1.1, one must determine the least-perimeter regions with prescribed area. For this purpose, the following strategy will be used: determine the regions with prescribed perimeter and biggest area. In fact, it is enough to show that if a region has the maximum area among all regions with a prescribed perimeter, then it has the minimum perimeter among all regions with the same prescribed area (see Lemma 4.1 below). Since all possible isoperimetric solutions beside the section were just listed, Lemma 4.1 will then refer to the above case 2. The other cases are proved analogously. Without loss of generality suppose that the geodesic halfdisk above $\{y = 1\}$ has maximum area when compared to any geodesic halfdisk below $\{y = c\}$ with the same perimeter.

Lemma 4.1. *Let Ω_0 be the geodesic halfdisk above $\{y = 1\}$ with $|\Omega_0| \geq |\Omega|$, whenever $|\partial\Omega| = |\partial\Omega_0|$, for any geodesic halfdisk Ω below $\{y = c\}$, $c > 1$. If Ω_1 is a geodesic halfdisk below $\{y = c\}$ with $|\Omega_0| = |\Omega_1|$, then $|\partial\Omega_0| \leq |\partial\Omega_1|$.*

Proof. Suppose by contradiction that $|\partial\Omega_0| > |\partial\Omega_1|$. By (1) and (2) the radius of the Euclidean circle that represents Ω_1 can be increased to get a geodesic halfdisk Ω' such that $|\partial\Omega'| = |\partial\Omega_0|$. This procedure could fail if Ω' surpassed $\{y = 1\}$, but then the section will prevail as the isoperimetric

solution. This fact will be proved later on in Section 5. By (2), the area increases with the radius. Therefore, $|\Omega'| > |\Omega_1| = |\Omega_0|$ and $|\partial\Omega'| = |\partial\Omega_0|$. This is a contradiction with the fact that Ω_0 maximizes the area when compared to regions of the same perimeter, by hypothesis. \square

Throughout this section the area of the possible isoperimetric solutions for a prescribed perimeter will be compared. All results here are proved in [4], in a style different from that of classical \mathbb{H}^2 -geometry (see [1, 7, 11], for example). There the authors apply the Gauss-Bonnet theorem to get an isoperimetric inequality between the area and the perimeter, whereas in [4] trigonometry is used. This offers a nice alternative for computing values in \mathbb{H}^2 , but is not reproduced here for the sake of brevity.

For Case 1 described above, the area of a geodesic halfdisk above $\{y = 1\}$ is compared with the area of a geodesic disk entirely contained in \mathcal{F}_c , when they have the same perimeter. Let \mathcal{S} be the Euclidean circle with radius r_2 , $0 < r_2 < y_2 - 1$, and center $(0, y_2)$, $1 < y_2 < c$, which delimits the geodesic halfdisk (see Figure 3 left).

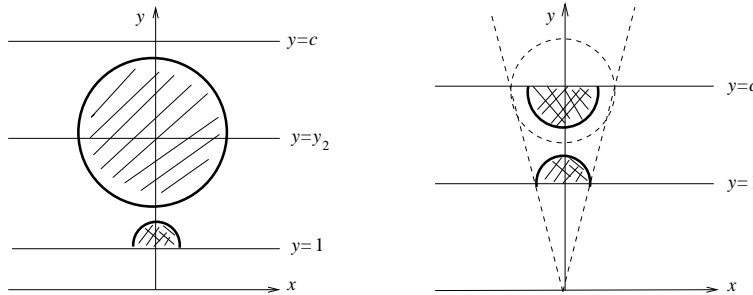


FIGURE 3. Cases 1 (left) and 2 (right).

Recall that the regions S_1^+ , S_2^- , H and E were defined in Section 3. The next lemma shows that $|S_1^+| > |S|$ when $|\partial S_1^+| = |\partial S|$.

Lemma 4.2. *Let $\theta_1, \theta_2 \in]0, \pi/2[$ such that*

$$\theta_1 \cot \theta_1 = \pi \cot \theta_2.$$

Then

$$\frac{2\theta_1}{\sin \theta_1} + 2 \cos \theta_1 - \pi > \frac{2\pi}{\sin \theta_2} - 2\pi.$$

Case 2 is related to Lemma 4.3, which shows that $|S_1^+| > |S_2^-|$ when $|\partial S_1^+| = |\partial S_2^-|$ (see Figure 3 right).

Lemma 4.3. *Let $\theta_1, \theta_2 \in]0, \pi/2[$ such that*

$$\theta_1 \cot \theta_1 = (\pi - \theta_2) \cot \theta_2.$$

Then

$$\frac{\theta_1}{\sin \theta_1} + \cos \theta_1 > \frac{\pi - \theta_2}{\sin \theta_2} - \cos \theta_2.$$

The next lemma shows that $|H| > |S_2^-|$ when $|\partial H| = |\partial S_2^-|$. Case 3 is illustrated in Figure 4 left.

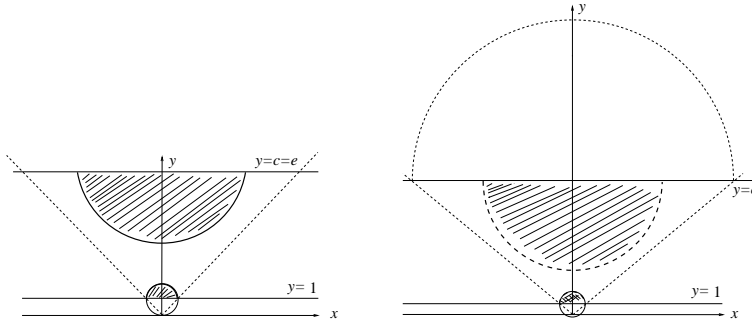


FIGURE 4. Cases 3 (left) and 4 (right).

Lemma 4.4. *Let $\theta_2 \in]0, \pi/2[$ such that $1 = (\pi - \theta_2) \cot \theta_2$. Then $2 > \frac{\pi - \theta_2}{\sin \theta_2} - \cos \theta_2$.*

Case 4 is related to the next lemma, which shows that $|E| > |S_2^-|$ when $|\partial E| = |\partial S_2^-|$ (see Figure 4 right).

Lemma 4.5. *Let $\alpha, \theta_2 \in]0, \pi/2[$ such that*

$$\frac{1}{\cos \alpha} \ln \left(\frac{1}{\sin \alpha} + \cot \alpha \right) = (\pi - \theta_2) \cot \theta_2.$$

Then

$$\frac{1}{\sin \alpha} + \frac{1}{\cot \alpha} \ln \left(\frac{1}{\sin \alpha} + \cot \alpha \right) \geq \frac{\pi - \theta_2}{\sin \theta_2} - \cos \theta_2.$$

From Lemmas 4.2 to 4.5, it is possible to conclude that the family of geodesic, horocycle and equidistant halfdisks above $\{y = 1\}$ are the solutions to the isoperimetric problem, instead of the geodesic halfdisks below $\{y = c\}$, $c > 1$.

5. ISOPERIMETRIC PROFILE IN \mathbb{R}_+^2

In this section the isoperimetric profile for \mathcal{F}_c (see Figure 5) is studied. A well-known result from Isoperimetric Problem Theory can be adapted to guarantee that the boundaries of the connected components of an isoperimetric solution are curves with the same constant geodesic curvature (for instance, see [1, Lemma 2.1]). It will be proved that a connected component of an isoperimetric region *must* be either a section or a halfdisk above the horocycle $\{y = 1\}$ before showing that a minimizing region is made of a single connected component. Here the analysis done in Section 3 of [4] will be needed. The perimeter of the section in \mathcal{F}_c is equal to $2 \ln c$. Now there are only three possibilities which are classified according to the hyperbolic distance $d = \ln c$: $d < 1$, $d = 1$, and $d > 1$. Only the analysis for the most general case $d > 1$ will be presented here, because the others are quite analogous.

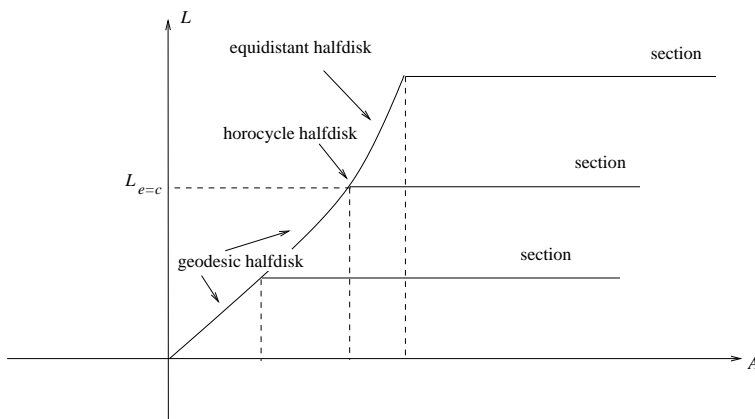


FIGURE 5. Isoperimetric profile for the region between the parallel horocycles.

Case $d > 1$.

Consider a horocycle $\{y = c\}$ with $c > e$. Let $A_0(c) = 4 - \pi$ be the area of the horocycle halfdisk S_0 above $\{y = 1\}$, centered at $(0, 1)$ with Euclidean radius $r_0(c) = 1$ and $|\partial S_0| = 2$. Let T_0 be a section with $|T_0| = A_0(c)$ and $A_1(c)$ be the area of an equidistant halfdisk S_1 above $\{y = 1\}$, centered at $(0, 1)$ with Euclidean radius $r_1(c)$ and $|\partial S_1| = |\partial T_1|$, where T_1 is a section with $|T_1| = A_1(c)$ (see Figure 6). In this case, observe that $|\partial T_1| > 2$.

Consequently,

- if $A = A_0(c) = 4 - \pi$ then $|S_0| = |T_0| = A$, but $|\partial T_0| > 2 = |\partial S_0|$.
Therefore, the minimizing Ω is a horocycle halfdisk;

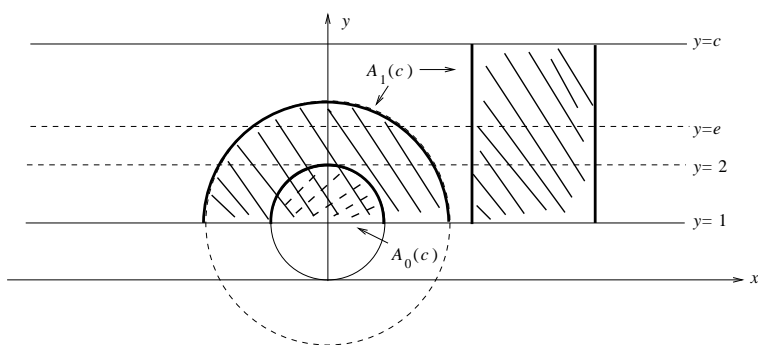


FIGURE 6. Case $c > e$.

- if $A = A_1(c)$ then $|S_1| = |T_1| = A$ and $|\partial S_1| = |\partial T_1|$. Therefore, the minimizing Ω is an equidistant halfdisk or a section;
- if $A < A_0(c)$, let S_2 be a geodesic halfdisk with $|S_2| = A$, centered at $(0, 1)$ and with Euclidean radius r_2 . Then $r_2 < r_0(c)$ and $|\partial S_2| < |\partial S_0|$. Let T_2 be a section with $|T_2| = A$. Then $|S_2| = |T_2| = A$, but $|\partial S_2| < |\partial T_2| = |\partial T_0| = |\partial S_0|$. Therefore, the minimizing Ω is a geodesic halfdisk;
- if $A_0(c) < A < A_1(c)$, let S_3 be an equidistant halfdisk with $|S_3| = A$, centered at $(0, 1)$ and with Euclidean radius r_3 . Then $r_0(c) < r_3 < r_1(c)$ and $|\partial S_3| < |\partial S_1|$. Let T_3 be a section with $|T_3| = A$. Then $|S_3| = |T_3| = A$, but $|\partial S_3| < |\partial T_3| = |\partial T_1| = |\partial S_1|$. Therefore, the minimizing Ω is an equidistant halfdisk;
- if $A > A_1(c)$, let S_4 be an equidistant halfdisk with $|S_4| = A$, centered at $(0, 1)$ and with Euclidean radius r_4 . Then $r_4 > r_1(c)$ and $|\partial S_4| > |\partial S_1|$. Let T_4 be a section with $|T_4| = A$. Then $|S_4| = |T_4| = A$, but $|\partial S_4| > |\partial T_4| = |\partial T_1| = |\partial S_1|$. Therefore, the minimizing Ω is a section.

Remark 5.1. A minimizing region consists of only one connected component, and in fact it is enough to show that it cannot have two. If this were the case, their geodesic curvatures would agree. Consider $A > 0$ and Ω' a region with area A and two disjoint sections. Their "gluing" would result in another section with area A but with smaller perimeter, because two vertical geodesics would not count anymore. Then Ω' is not minimizing.

The other case to consider is two connected components consisting of two geodesic halfdisks above $\{y = 1\}$. In this case, the fact that a non-regular region is not minimizing can be used: let $A > 0$ and Ω' be a region with area A and two geodesic halfdisks above $\{y = 1\}$ with the same Euclidean radius,

hence the same geodesic curvature. By sliding one of them over $\{y = 1\}$ until it touches the other, since horizontal translations are isometries of the hyperbolic plane, a non-regular region Ω'' with area A is obtained. Then Ω'' does not have the least-perimeter among all regions with prescribed area A . Since $|\Omega'| = |\Omega''|$, Ω' is not minimizing.

Therefore, a minimizing region must consist of a single connected component.

Theorem 1.1 is proved now.

Proof. The first part of Theorem 1.1 was already discussed in Remark 2.1. The existence of such an isoperimetric region follows from adaptations of some results from [8] and [9]: the group G of isometries of \mathbb{R}_+^2 that leave \mathcal{F}_c invariant consists of horizontal Euclidean translations and Euclidean reflections with respect to a vertical geodesic, so that \mathcal{F}_c/G is homeomorphic to the interval $[0, 1]$, hence compact.

The second part of Theorem 1.1 follows from the analysis of the isoperimetric profile done in the three possibilities above, together with Remark 5.1. \square

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ISOPERIMETRIC REGIONS IN THE HYPERBOLIC PLANE

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