EXISTENCE OF NONOSCILLATORY SOLUTIONS OF HIGHER-ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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ABSTRACT. This paper studies the existence of nonoscillatory solutions of a higher-order nonlinear neutral delay difference equation

$$\Delta\Big(a_{kn}\cdots\Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n+b_nx_{n-d}))\big)\Big) + f\big(n,x_{n-r_{1n}},x_{n-r_{2n}},\dots,x_{n-r_{sn}}\big)=0,\quad n\ge n_0,$$

where $n_0\ge 0, n\ge 0, d>0, k>0, s>0$ are integers, $\{a_{in}\}_{n\ge n_0}(i=1,2,\dots,k)$ and $\{b_n\}_{n\ge n_0}$ are real sequences, $f\colon\{n:n\ge n_0\}\times\mathbb{R}^s\to\mathbb{R}$ is a mapping and $\bigcup_{j=1}^{s}\{r_{jn}\}_{n\ge n_0}\subseteq\mathbb{Z}$. By applying Krasnoselskii's Fixed Point Theorem, some sufficient conditions for the existence of nonoscillatory solutions of this equation are established and indicated through five theorems according to the range of value of the sequence b_n.

1. Introduction and preliminaries

In recent years, more and more people are interested in the study of the solvability of difference equations; see [1–17] and the references cited therein. Many authors have paid attention to various difference equations. For example,

$$\Delta(a_n\Delta x_n)+p_nx_{g(n)}=0,\quad n\ge 0,\qquad (1.1)\quad ([14])$$

$$\Delta(a_n\Delta x_n)=q_nx_{n+1},\quad \Delta(a_n\Delta x_n)=q_nf(x_{n+1}),\quad n\ge 0,\qquad (1.2)\quad [11]$$

$$\Delta^2(x_n+px_{n-m})+p_nx_{n-k}-q_nx_{n-l}=0,\quad n\ge n_0,\qquad (1.3)\quad [6]$$

$$\Delta^2(x_n+px_{n-k})+f(n,x_n)=0,\quad n\ge 1,\qquad (1.4)\quad [10]$$

$$\Delta^2(x_n-px_{n-\tau})=\sum_{i=1}^{m}q_if_i(x_{n-\sigma_i}),\quad n\ge n_0,\qquad (1.5)\quad [9]$$

$$\Delta(a_n\Delta(x_n+bx_{n-\tau}))+f(n,x_{n-d_{1n}},x_{n-d_{2n}},\dots,x_{n-d_{kn}})=c_n,\quad n\ge n_0,\qquad (1.6)\quad [8]$$

$$\Delta^m(x_n+cx_{n-k})+p_nx_{n-r}=0,\quad n\ge n_0,\qquad (1.7)\quad [15]$$

$$\Delta^m(x_n+c_nx_{n-k})+p_nf(x_{n-r})=0,\quad n\ge n_0,\qquad (1.8)\quad [3,4,12,13]$$

$$\Delta^m(x_n + cx_{n-k}) + \sum_{s=1}^{u} p_n^s f_s(x_{n-r_s}) = q_n, \quad n \geq n_0, \qquad (1.9) \quad ([16])$$

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \quad n \geq n_0. \qquad (1.10) \quad ([17])$$

Motivated and inspired by the papers mentioned above, in this paper we consider the following higher-order nonlinear neutral delay difference equation

$$\begin{aligned} &\Delta\Big(a_{kn} \cdots \Delta\big(a_{2n}\Delta(a_{1n}\Delta(x_n + b_n x_{n-d}))\big)\Big) \\ &\qquad + f\big(n, x_{n-r_{1n}}, x_{n-r_{2n}}, \ldots, x_{n-r_{sn}}\big) = 0, \quad n \geq n_0, \end{aligned} \qquad (1.11)$$

where $n_0 \geq 0$, $n \geq 0$, $d > 0$, $k > 0$, $s > 0$ are integers, $\{a_{in}\}_{n \geq n_0} (i = 1, 2, \ldots, k)$ and $\{b_n\}_{n \geq n_0}$ are real sequences, $f \colon \{n : n \geq n_0\} \times \mathbb{R}^s \to \mathbb{R}$ is a mapping and $\bigcup_{j=1}^{s} \{r_{jn}\}_{n \geq n_0} \subseteq \mathbb{Z}$. However, to the authors' knowledge, few papers in the literature can be found dealing with the existence of nonoscillatory solutions for (1.11), which is mainly due to the technical difficulties arising in its analysis, especially when there are more than two factors a_{in} before each forward difference Δ. Clearly, difference equations (1.1)–(1.10) are special cases of (1.11). By using Krasnoselskii's Fixed Point Theorem, the existence of nonoscillatory solutions of (1.11) is established.

Lemma 1.1 (**Krasnoselskii's Fixed Point Theorem**). *Let Ω be a bounded closed convex subset of a Banach space X and $T_1, T_2 \colon \Omega \to X$ satisfy $T_1 x + T_2 y \in \Omega$ for each $x, y \in \Omega$. If T_1 is a contraction mapping and T_2 is a completely continuous mapping, then the equation $T_1 x + T_2 x = x$ has at least one solution in Ω.*

The forward difference Δ is defined as usual, i.e., $\Delta x_n = x_{n+1} - x_n$. The higher-order difference for a positive integer m is defined as $\Delta^m x_n = \Delta(\Delta^{m-1} x_n), \Delta^0 x_n = x_n$. Throughout this paper, assume $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ stand for the sets of all positive integers and integers, respectively, $\alpha = \inf\{n - r_{jn} : 1 \leq j \leq s, n \geq n_0\}$, $\beta = \min\{n_0 - d, \alpha\}$, $\lim_{n \to \infty}(n - r_{jn}) = +\infty, 1 \leq j \leq s$, l_β^∞ denotes the set of real sequences defined on the set of positive integers larger than β where any individual sequence is bounded with respect to the usual supremum norm $||x|| = \sup_{n \geq \beta} |x_n|$ for $x = \{x_n\}_{n \geq \beta} \in l_\beta^\infty$. It is well-known that l_β^∞ is a Banach space under the supremum norm. A subset Ω of a Banach space X is relatively compact if every sequence in Ω has a subsequence converging to an element of X.

Definition 1.1 ([**5**]). *A set Ω of sequences in l_β^∞ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exists an integer N_0 such that*

$$|x_i - x_j| < \varepsilon,$$

for all $i, j > N_0, x = x_k \in \Omega$.

Lemma 1.2 (**Discrete Arzela-Ascoli's Theorem** [**5**]). *A bounded, uniformly Cauchy subset Ω of l_β^∞ is relatively compact.*

Let

$$A(M,N) = \big\{x = \{x_n\}_{n\geq\beta} \in l_\beta^\infty : M \leq x_n \leq N, n \geq \beta\big\} \quad \text{for} \quad N > M > 0.$$

Obviously, $A(M,N)$ is a bounded closed and convex subset of l_β^∞. Put

$$\overline{b} = \limsup_{n\to\infty} b_n \quad \text{ and } \quad \underline{b} = \liminf_{n\to\infty} b_n.$$

By a solution of (1.11), we mean a sequence $\{x_n\}_{n\geq\beta}$ with a positive integer $N_0 \geq n_0 + d + |\alpha|$ such that (1.11) is satisfied for all $n \geq N_0$. As is customary, a solution of (1.11) is said to be oscillatory about zero, or simply oscillatory if the terms x_n of the sequence $\{x_n\}_{n\geq\beta}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

2. Existence of nonoscillatory solutions

In this section, five existence results of nonoscillatory solutions of (1.11) are offered.

Theorem 2.1. *Assume that there exist constants b, M and N with $N > M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$ such that for $n \geq n_0$*

$$|b_n| \leq b < \frac{N-M}{2N}, \tag{2.1}$$

$$\begin{aligned}\big|f(n,u_1,u_2,\ldots,u_s) - f(n,v_1,v_2,\ldots,v_s)\big| \\ \leq h_n \max\big\{|u_i - v_i| : u_i, v_i \in [M,N], 1 \leq i \leq s\big\},\end{aligned} \tag{2.2}$$

$$\big|f(n,u_1,u_2,\ldots,u_s)\big| \leq q_n,\ u_i \in [M,N], 1 \leq i \leq s, \tag{2.3}$$

$$\sum_{t=n_0}^{\infty} \max\left\{\frac{1}{|a_{it}|}, h_t, q_t : 1 \leq i \leq k\right\} < +\infty. \tag{2.4}$$

Then (1.11) has a nonoscillatory solution in $A(M,N)$.

Proof. Choose $L \in (M + bN, N - bN)$. By (2.1) and (2.4), an integer $N_0 > n_0 + d + \alpha$ can be chosen such that

$$|b_n| \le b < \frac{N - M}{2N}, \quad \text{for all} \quad n \ge N_0, \tag{2.5}$$

and

$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \le \min\{L - bN - M, N - bN - L\}. \tag{2.6}$$

Define two mappings $T_1, T_2 \colon A(M,N) \to X$ by

$$(T_1x)_n = \begin{cases} L - b_n x_{n-d}, & n \ge N_0; \\ (T_1x)_{N_0}, & \beta \le n < N_0. \end{cases} \tag{2.7}$$

$$(T_2x)_n = \begin{cases} (-1)^k \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, n \ge N_0, \\ (T_2x)_{N_0}, \ \beta \le n < N_0, \end{cases} \tag{2.8}$$

for all $x \in A(M,N)$.

(i) We claim that $T_1x + T_2y \in A(M,N)$, for all $x, y \in A(M,N)$.

In fact, for every $x, y \in A(M,N)$ and $n \ge N_0$, it follows from (2.3), (2.5) and (2.6) that

$$\begin{aligned} &(T_1x + T_2y)_n \\ &\ge L - bN - \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{|f(t, y_{t-r_{1t}}, y_{t-r_{2t}}, \ldots, y_{t-r_{st}})|}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\ge L - bN - \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\ge M \end{aligned}$$

and

$$\begin{aligned} &(T_1x + T_2y)_n \\ &\le L + bN + \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \le N. \end{aligned}$$

That is, $(T_1x + T_2y)(A(M,N)) \subseteq A(M,N)$.

(ii) We claim that T_1 is a contraction mapping on $A(M,N)$.

In reality, for any $x, y \in A(M,N)$ and $n \ge N_0$, it is easy to derive that

$$\big|(T_1x)_n - (T_1y)_n\big| \le |b_n||x_{n-d} - y_{n-d}| \le b||x - y||,$$

which implies that

$$||T_1x - T_1y|| \leq b||x - y||.$$

$b < \frac{N-M}{2N} < 1$ ensures that T_1 is a contraction mapping on $A(M, N)$.

(iii) We claim that T_2 is completely continuous.

We first show that T_2 is continuous. Let $x^{(u)} = \{x_n^{(u)}\} \in A(M, N)$ be a sequence such that $x_n^{(u)} \to x_n$ as $u \to \infty$. Since $A(M, N)$ is closed, $x = \{x_n\} \in A(M, N)$. For $n \geq N_0$, (2.2) guarantees that

$$\begin{aligned}
&\left|T_2 x_n^{(u)} - T_2 x_n\right| \\
&\quad\leq \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \\
&\qquad \frac{\left|f(t, x^{(u)}_{t-r_{1t}}, x^{(u)}_{t-r_{2t}}, \ldots, x^{(u)}_{t-r_{st}}) - f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\quad\leq \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{h_t \max\{|x^{(u)}_{t-r_{jt}} - x_{t-r_{jt}}| : 1 \leq j \leq s\}}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\quad\leq \|x^{(u)} - x\| \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{h_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|}.
\end{aligned}$$

This inequality and (2.4) imply that T_2 is continuous.

Next we show $T_2 A(M, N)$ is relatively compact. By (2.4), for any $\varepsilon > 0$, take $N_1 \geq N_0$ large enough so that

$$\sum_{t_1=N_1}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} < \frac{\varepsilon}{2}. \tag{2.9}$$

Then, for any $x = \{x_n\} \in A(M, N)$ and $n_1, n_2 \geq N_1$, (2.9) ensures that

$$\begin{aligned}
&\left|T_2 x_{n_1} - T_2 x_{n_2}\right| \\
&\quad\leq \sum_{t_1=n_1}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left|f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\qquad + \sum_{t_1=n_2}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left|f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\quad\leq \sum_{t_1=N_1}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\qquad + \sum_{t_1=N_1}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which means $T_2A(M,N)$ is uniformly Cauchy. Therefore, by Lemma 1.2, $T_2A(M,N)$ is relatively compact. By Lemma 1.1, there exists $x = \{x_n\} \in A(M,N)$ such that $T_1x + T_2x = x$, which is a bounded nonoscillatory solution of (1.11). This completes the proof. $\square$

Theorem 2.2. *Assume that there exist constants M and N with $N > \frac{2-\underline{b}}{1-\overline{b}}M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, satisfying (2.2)–(2.4) and*

$$b_n \geq 0, \ \text{and } 0 \leq \underline{b} \leq \overline{b} < 1. \tag{2.10}$$

Then (1.11) has a nonoscillatory solution in $A(M,N)$.

Proof. Choose $L \in (M + \frac{1+\overline{b}}{2}N, N + \frac{\underline{b}}{2}M)$. By (2.10) and (2.4), an integer $N_0 > n_0 + d + \alpha$ can be chosen such that

$$\frac{\underline{b}}{2} \leq b_n \leq \frac{1+\overline{b}}{2}, \quad \text{for all} \quad n \geq N_0, \tag{2.11}$$

and

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\leq \min\Big\{L - M - \frac{1+\overline{b}}{2}N, N - L + \frac{\underline{b}}{2}M\Big\}. \end{aligned} \tag{2.12}$$

Define two mappings $T_1, T_2 : A(M,N) \to X$ as (2.7) and (2.8). The rest of the proof is analogous to that in Theorem 2.1. This completes the proof. $\square$

Similar to the proof of Theorem 2.2, we have the following theorem.

Theorem 2.3. *Assume that there exist constants M and N with $N > \frac{2+\overline{b}}{1+\underline{b}}M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, satisfying (2.2)–(2.4) and*

$$b_n \leq 0, \ \text{and } -1 < \underline{b} \leq \overline{b} \leq 0. \tag{2.13}$$

Then (1.11) has a nonoscillatory solution in $A(M,N)$.

Theorem 2.4. *Assume that there exist constants M and N with $N > \frac{\underline{b}(\overline{b}^2-\underline{b})}{\overline{b}(\underline{b}^2-\overline{b})}M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, satisfying (2.2)–(2.4) and*

$$b_n > 1, \quad 1 < \underline{b} \ \text{and } \overline{b} < \underline{b}^2 < +\infty. \tag{2.14}$$

Then (1.11) has a nonoscillatory solution in $A(M,N)$.

Proof. Take $\varepsilon \in (0, \underline{b}-1)$ sufficiently small satisfying

$$1 < \underline{b} - \varepsilon < \overline{b} + \varepsilon < (\underline{b} - \varepsilon)^2 \tag{2.15}$$

and

$$\big((\overline{b}+\varepsilon)(\underline{b}-\varepsilon)^2 - (\overline{b}+\varepsilon)^2\big)N > \big((\overline{b}+\varepsilon)^2(\underline{b}-\varepsilon) - (\underline{b}-\varepsilon)^2\big)M. \tag{2.16}$$

Choose $L \in \big((\overline{b}+\varepsilon)M + \frac{\overline{b}+\varepsilon}{\underline{b}-\varepsilon}N, (\underline{b}-\varepsilon)N + \frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M\big)$. By (2.15) and (2.4), an integer $N_0 > n_0 + d + \alpha$ can be chosen such that

$$\underline{b} - \varepsilon < b_n < \overline{b} + \varepsilon, \quad \text{ for all } b \geq N_0, \tag{2.17}$$

and

$$\begin{aligned} &\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{|\prod_{i=1}^{k} a_{it_i}|} \\ &\quad \leq \min\left\{ \frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}L - (\underline{b}-\varepsilon)M - N, \frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon}M + (\underline{b}-\varepsilon)N - L \right\}. \end{aligned} \tag{2.18}$$

Define two mappings $T_1, T_2 \colon A(M,N) \to X$ by

$$(T_1x)_n = \begin{cases} \frac{L}{b_{n+d}} - \frac{x_{n+d}}{b_{n+d}}, & n \geq N_0; \\ (T_1x)_{N_0}, & \beta \leq n < N_0. \end{cases} \tag{2.19}$$

$$(T_2x)_n = \begin{cases} \frac{(-1)^k}{b_{n+d}} \sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, n \geq N_0 \\ (T_2x)_{N_0},\ \beta \leq n < N_0 \end{cases} \tag{2.20}$$

for all $x \in A(M,N)$. The rest of the proof is analogous to that in Theorem 2.1. This completes the proof. □

Similar to the proof of Theorem 2.4, we have the following theorem.

Theorem 2.5. *Assume that there exist constants M and N with $N > \frac{1+\underline{b}}{1+\overline{b}}M > 0$ and sequences $\{a_{in}\}_{n \geq n_0} (1 \leq i \leq k)$, $\{b_n\}_{n \geq n_0}$, $\{h_n\}_{n \geq n_0}$, $\{q_n\}_{n \geq n_0}$, satisfying (2.2)–(2.4) and*

$$b_n < -1, \quad -\infty < \underline{b} \text{ and } \overline{b} < -1. \tag{2.21}$$

Then (1.11) has a nonoscillatory solution in $A(M,N)$.

Remark 2.1. *Theorems 2.1–2.5 extend and improve Theorem 1 of Cheng [6] and Theorems 2.3–2.7 of Liu, Xu and Kang [8].*

3. An example

In this section, an example is given to illustrate the advantage of the above result.

Example 3.1. *Consider the following third-order nonlinear neutral delay difference equation:*

$$\begin{aligned} &\Delta\Big((2^n - n)\Delta\big((n^2 - n + 1)\Delta(x_n + \frac{3^n - 2}{4^n}x_{n-4})\big)\Big) \\ &\quad + \frac{\cos(2x_{n-2})}{n^2} - \frac{\sin(3x_{n-3})}{n^3} = 0, \quad n \geq 5, \end{aligned} \tag{3.1}$$

where

$$a_{1n} = n^2 - n + 1, \quad a_{2n} = 2^n - n, \quad b_n = \frac{3^n - 2}{4^n},$$

$$f(n, u_1, u_2) = \frac{\cos(2u_1)}{n^2} - \frac{\sin(3u_2)}{n^3}, \quad h_n = q_n = \frac{2}{n^2}.$$

Choose $M = 1$ and $N = 5$. It can be verified that the assumptions of Theorem 2.2 are fulfilled. It follows from Theorem 2.2 that (3.1) *has a nonoscillatory solution in $A(1, 5)$.*

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