HAMILTONIAN-CONNECTED GRAPHS WITH LARGE NEIGHBORHOODS AND DEGREES

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Abstract. For a simple graph G, let $NC(G) = \min\{|N(u) \cup N(v)| : u, v \in V(G),\ uv \notin E(G)\}$. In this paper we prove that if $NC(G) + \delta(G) \geq |V(G)|$, then either G is Hamiltonian-connected, or G belongs to a well-characterized class of graphs. The former result by Dirac, Ore and Faudree et al. are extended.

1. Introduction

Graphs considered in this paper are finite and simple. Undefined notations and terminologies can be found in [1]. In particular, we use $V(G)$, $E(G)$, $\kappa(G)$, $\delta(G)$, and $\alpha(G)$ to denote *the vertex set*, *the edge set*, *the connectivity*, *the minimum degree* and *the independence number* of G, respectively. If G is a graph and $u, v \in V(G)$, then a path in G from u to v is called a (u, v)-path of G. If $v \in V(G)$ and H is a subgraph of G, then $N_H(v)$ denotes the set of vertices in H that are adjacent to v in G. Thus, $d_H(v)$, the degree of v relative to H, is $|N_H(v)|$. We also write $d(v)$ for $d_G(v)$ and $N(v)$ for $N_G(v)$. If C and H are subgraphs of G, then $N_C(H) = \cup_{u \in V(H)} N_C(u)$, and $G - C$ denotes the subgraph of G induced by $V(G) - V(C)$. For vertices $u, v \in V(G)$, the distance between u and v, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G, or ∞ if no such path exists. Let $P_m = x_1 x_2 \cdots x_m$ denote a path of order m. Define $N^+_{P_m}(u) = \{x_{i+1} \in V(P_m) : x_i \in N_{P_m}(u)\}$ and $N^-_{P_m}(u) = \{x_{i-1} \in V(P_m) : x_i \in N_{P_m}(u)\}$. That means if $x_1 \in N_{P_m}(u)$, then $|N^-_{P_m}(u)| = |N_{P_m}(u)| - 1$ and if $x_m \in N_{P_m}(u)$, then $|N^+_{P_m}(u)| = |N_{P_m}(u)| - 1$.

For a graph G, define $NC(G) = \min\{|N(u) \cup N(v)| : u, v \in V(G), uv \notin E(G)\}$ and $NCD(G) = \min\{|N(u) \cup N(v)| + d(w) : u, v, w \in V(G), uv \notin E(G), wv \text{ or } wu \notin E(G)\}$.

Let G and H be two graphs. We use $G \cup H$ to denote the disjoint union of G and H and $G \bigvee H$ to denote the graph obtained from $G \cup H$ by joining every vertex of G to every vertex of H. We use K_n and K_n^c to denote the complete graph on n vertices and the empty graph on n vertices, respectively. Let G_n denote the family of all simple graphs of order n. For notational convenience, we also use G_n to denote a simple graph of order n. As an example, $G_2 \in \{K_2, K_2^c\}$. Define $G_2 : G_n$ to be the family of 2-connected graphs each of which is obtained from $G_2 \cup G_n$ by joining every vertex of G_2 to some vertices of G_n so that the resulting graph G satisfies $NCD(G) \geq |V(G)| = n + 2$. For notational convenience, we also use $G_2 : G_n$ to denote a member in the family.

A graph G is *Hamiltonian* if it has a spanning cycle, and *Hamiltonian-connected* if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path. There have been intensive studies on sufficient degree and/or neighborhood union conditions for Hamiltonian graphs and Hamiltonian-connected graphs. The following is a summary of these results that are related to our study.

Theorem 1.1. *Let G be a simple graph on n vertices.*

(i) *(Dirac, [2]) If $\delta(G) \geq n/2$, then G is Hamiltonian.*
(ii) *(Ore, [3]) If $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*
(iii) *(Faudree et al., [5]) If G is 3-connected, and if $NC(G) \geq (2n+1)/3$, then G is Hamiltonian-connected.*
(iv) *(Faudree et al., [6]) If G is 2-connected, and if $NC(G) + \delta(G) \geq n$, then G is Hamiltonian.*
(v) *(Wei, [7]) If G is a 2-connected, and if $\min\{d(u) + d(v) + d(w) - |N(u) \cap N(v) \cap N(w)| : u, v, w \in V(G), uv, vw, wu \notin E(G)\} \geq n+1$, then G is Hamiltonian-connected with some well characterized exceptional graphs.*

Motivated by the results above, this paper aims to investigate the Hamiltonian and Hamiltonian-connected properties of graphs with relatively large $NCD(G)$. The main theorem is the following.

Theorem 1.2. *If G is a 2-connected graph with n vertices and if $NC(G) + \delta(G) \geq n$, then one of the following must hold:*

(i) *G is Hamiltonian-connected,*
(ii) *$G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K^c_{n/2}, G_2 : (K_s \cup K_h \cup K_t), G_3 \bigvee (K_{(n-3)/3} \cup K_{(n-3)/3} \cup K_{(n-3)/3})\}$.*

Let $G = G_2 : (K_s \cup K_h \cup K_t)$, and let x be a vertex in K_s and y a vertex in K_h. Then $d(x) + d(y) < |V(G)|$. Also, $G_3 \bigvee (K_s \cup K_h \cup K_t)$ satisfies the condition that $d(x) + d(y) \geq n$ for any two nonadjacent vertices x, y if and only if $s = h = t = 1$. Thus, Corollary 1.3 below follows from Theorem 1.2 immediately and it extends Theorem 1.1(ii).

Corollary 1.3. *If G is a graph of order n satisfying $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian-connected or $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K^c_{n/2}\}$.*

Since none of $G_2 : (K_s \cup K_h)$, $G_{n/2} \bigvee K^c_{n/2}$, $G_2 : (K_s \cup K_h \cup K_t)$, and $G_3 \bigvee (K_s \cup K_h \cup K_t)$ satisfies the condition that $d(x) + d(y) \geq n + 1$ for every pair of nonadjacent vertices x, y, Theorem 1.2 also implies the following result of Ore [4].

Corollary 1.4 (Ore, [4])**.** *If G is a 2-connected graph of order n satisfying $d(x)+d(y) \geq n+1$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian-connected.*

As $G_2 : (K_s \cup K_h), G_{n/2} \bigvee K^c_{n/2}$, and $G_3 \bigvee (K_s \cup K_h \cup K_t)$ are all Hamiltonian, Theorem 1.2 implies Theorem 1.5.

Theorem 1.5. *If G is a 2-connected graph with n vertices such that $NCD(G) \geq n$, then G is Hamiltonian.*

Clearly, we have $NCD(G) \geq NC(G)+\delta(G)$. Thus, if $NC(G)+\delta(G) \geq n$, we have $NCD(G) \geq n$. And clearly if $\max\{s, h, t\} \neq \min\{s, h, t\}$, then $NC(G) + \delta(G)$ of $G_3 \bigvee (K_S \cup K_h \cup K_t)$ must be less than or equal to $n - 1$. Thus, Theorem 1.5 implies the following result of Hamilton-connected graph under Faudree et al. condition.

Corollary 1.6. *If G is a 2-connected graph with n vertices such that $NC(G) + \delta(G) \geq n$, then G is Hamiltonian-connected or $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K^c_{n/2}, G_2 : (K_s \cup K_h \cup K_t), G_3 \bigvee (K_{(n-3)/3} \cup K_{(n-3)/3} \cup K_{(n-3)/3})\}$.*

2. Proof of Corollary 1.6.

For a path $P_m = x_1x_2 \cdots x_m$, we use $[x_i, x_j]$ to denote the section $x_ix_{i+1} \cdots x_j$ of the path P_m if $i < j$, and to denote the section $x_ix_{i-1} \cdots x_j$ of the path P_m if $i > j$. For notational convenience, we also use $[x_i, x_j]$ to denote the vertex set of this path. If P_1 is an (x, y)-path and P_2 is a (y, z)-path in a graph G such that $V(P_1) \cap V(P_2) = \{y\}$, then P_1P_2 denotes the (x, z)-path of G induced by $E(P_1) \cup E(P_2)$.

Let G be a 2-connected graph on n vertices such that

$$NCD(G) \geq n. \tag{1}$$

We shall assume that G is not Hamiltonian-connected to show that Theorem 1.2(ii) must hold. Thus, there exist $x, y \in V(G)$ such that G does not have a spanning (x, y)-path. Let

$$P_m = x_1x_2 \cdots x_m \text{ be a longest } (x, y)\text{-path in } G, \tag{2}$$

where $x_1 = x$ and $x_m = y$. Since P_m is not a Hamilton-path, $G - P_m$ has at least one component.

Lemma 2.1. *Suppose that H is a component of $G - P_m$. Then each of the following holds.*

(i) *For all i with $1 < i < m$, if $x_i \in N_{P_m}(H) \setminus \{x_1, x_m\}$, then $x_{i+1} \notin N_{P_m}(H)$ and $x_{i-1} \notin N_{P_m}(H)$; if $x_1 \in N_{P_m}(H)$, then $x_2 \notin N_{P_m}(H)$, and if $x_m \in N_{P_m}(H)$, then $x_{m-1} \notin N_{P_m}(H)$.*

(ii) *If $x_i, x_j \in N_{P_m}(H)$ with $1 \le i < j < m$, then $x_{i+1}x_{j+1} \notin E(G)$; if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \le m$, then $x_{i-1}x_{j-1} \notin E(G)$. Consequently, both $N^+_{P_m}(H)$ and $N^-_{P_m}(H)$ are independent sets.*

(iii) *Let $x_i, x_j \in N_{P_m}(H)$ with $1 \le i < j < m$. If $x_t x_{j+1} \in E(G)$ for some vertex $x_t \in [x_{j+2}, x_m]$, then $x_{t-1}x_{i+1} \notin E(G)$ and $x_{t-1} \notin N_{P_m}(H)$; if $x_t x_{j+1} \in E(G)$ for some vertex $x_t \in [x_{i+1}, x_j]$, then $x_{t+1}x_{i+1} \notin E(G)$.*

(iii)' *Let $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \le m$. If $x_t x_{i-1} \in E(G)$ for some vertex $x_t \in [x_1, x_{i-2}]$, then $x_{t+1}x_{j-1} \notin E(G)$ and $x_{t+1} \notin N_{P_m}(H)$; if $x_t x_{i-1} \in E(G)$ for some vertex $x_t \in [x_{i+1}, x_j]$, then $x_{t-1}x_{j-1} \notin E(G)$.*

(iv) *If $x_i, x_j \in N_{P_m}(H)$ with $1 \le i < j < m$, then no vertex of $G - (V(P_m) \cup V(H))$ is adjacent to both x_{i+1} and x_{j+1}; if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \le m$, then no vertex of $G - (V(P_m) \cup V(H))$ is adjacent to both x_{i-1} and x_{j-1}.*

(v) *Suppose that $u \in V(H)$ and $\{x_1, x_m\} \subseteq N_{P_m}(u)$. If $x_i, x_j \in N_{P_m}(H)$ with $1 \le i < j < m$, then for any $v \in V(G) \setminus (N^+_{P_m}(H) \cup \{u\})$, $vx_{i+1} \in E(G)$ or $vx_{j+1} \in E(G)$; if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \le m$, then for any $v \in V(G) \setminus (N^-_{P_m}(H) \cup \{u\})$, $vx_{i-1} \in E(G)$ or $vx_{j-1} \in E(G)$.*

Proof. (i), (ii) and (iv) follow immediately from the assumption that P_m is a longest (x_1, x_m)-path in G. It remains to show that (iii) and (v) must hold. Since $x_i, x_j \in N_{P_m}(H)$, there exist $x'_i, x'_j \in V(H)$ such that $x_i x'_i, x_j x'_j \in E(G)$. Let P' denote an (x'_i, x'_j)-path in H.

(iii) Suppose that the first part of (iii) fails. Then there exists a vertex $x_t \in \{x_{j+2}, x_{j+3}, \ldots, x_m\}$ such that $x_t x_{j+1} \in E(G)$ and $x_{t-1}x_{i+1} \in E(G)$. Then $[x_1, x_i]P'\,[x_j, x_{i+1}]\,[x_{t-1}, x_{j+1}][x_t, x_m]$ is a longer (x_1, x_m)-path, contrary to (2). Hence, $x_t x_{j+1} \notin E(G)$. Next, we assume that x_{t-1} is adjacent to some vertex $x'_{t-1} \in V(H)$. Let P'' denote an (x'_{t-1}, x'_j)-path in H. Then $[x_1, x_j]P''[x_{t-1}, x_{j+1}][x_t, x_m]$ is a longer (x_1, x_m)-path, contrary to (2). The proof for (iii)' is similar, and so it is omitted.

(v) For vertices $x_i, x_j \in N_{P_m}(H)$ with $1 \le i < j < m$, by Lemma 2.1(i), we have $x_{i+1} \notin N(u)$, $x_{j+1} \notin N(u)$ and by Lemma 2.1(ii), we have $x_{i+1}x_{j+1} \notin E(G)$. Since $N^+_{P_m}(H)$ is an independent set, then $N(v_{i+1}) \cup N(v_{j+1}) \subseteq V(G) - N^+_{P_m}(H) \cup \{u\}$. Furthermore, $d(u) \le |N_{P_m}(H)| = |N^+_{P_m}(H) \cup \{u\}|$, and so we have $|N(v_{i+1}) \cup N(v_{j+1})| + d(u) \le |V(G)| - |N^+_{P_m}(H) \cup \{u\}| + d(u) \le n$. Since $x_{i+1}x_{j+1} \notin E(G), ux_{i+1} \notin E(G), ux_{j+1} \notin E(G)$, by (1), $|N(v_{i+1}) \cup N(v_{j+1})| + d(u) \ge n$ and so we have $N(v_{i+1}) \cup$

$N(v_{j+1}) = V(G) - N^+_{P_m}(H) \cup \{u\}$, which implies for all $v \in V(G) \setminus (N^+_{P_m}(H) \cup \{u\})$, $vx_{i+1} \in E(G)$ or $vx_{j+1} \in E(G)$. Similarly, if $x_i, x_j \in N_{P_m}(H)$ with $1 < i < j \leq m$, then for any $v \in V(G) \setminus (N^-_{P_m}(H) \cup \{u\})$, $vx_{i-1} \in E(G)$ or $vx_{j-1} \in E(G)$. This proves (v). □

Lemma 2.2. *Each of the following holds.*

(i) *If there is a component H of $G-P_m$ such that $N_{P_m}(H) = \{x_1, x_m\}$, then $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is a complete subgraph.*
(ii) *If $N_{P_m}(G-P_m) = \{x_1, x_m\}$, then $G-P_m$ has at most 2 components.*
(iii) *If $N_{P_m}(G-P_m) = \{x_1, x_m\}$, then every component of $G-P_m$ is a complete subgraph.*
(iv) *If $N_{P_m}(G-P_m) = \{x_1, x_m\}$, then $G \in \{G_2 : (K_s \cup K_h), G_2 : (K_s \cup K_h \cup K_t)\}$.*

Proof. (i) Suppose, to the contrary, that $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is not a complete subgraph. Then there exist $x_i, x_j \in \{x_2, x_3, \ldots x_{m-1}\}$ such that $x_ix_j \notin E(G)$. Since $N_{P_m}(G-P_m) = \{x_1, x_m\}$, then $(N(x_i) \cup N(x_j)) \cap (V(H) \cup \{x_i, x_j\}) = \emptyset$ and so $|N(x_i) \cup N(x_j)| \leq |V(G) \setminus V(H)| - |\{x_i, x_j\}|$. Let $u \in V(H)$. Then $ux_i \notin E(G)$ and $ux_j \notin E(G)$. Furthermore, we have $d(u) \leq |V(H) \setminus \{u\}| + |\{x_1, x_m\}|$, and so $|N(x_i) \cup N(x_j)| + d(u) \leq |V(G) \setminus V(H)| - |\{x_i, x_j\}| + |V(H) \setminus \{u\}| + |\{x_1, x_m\}| \leq n-1$, contrary to (1).

(ii) Suppose that $G-P_m$ has at least three components H_1, H_2, and H_3. Let $u \in V(H_1)$ and $v \in V(H_2)$. Then $uv \notin E(G)$. Since $N_{P_m}(G-P_m) = \{x_1, x_m\}$, then we have $ux_2 \notin E(G), vx_2 \notin E(G)$. Again by $N_{P_m}(G-P_m) = \{x_1, x_m\}$, we have $N(u) \cup N(v) \subseteq (V(H_1)-\{u\}) \cup (V(H_2)-\{v\}) \cup \{x_1, x_m\}$ and $N(x_2) \subseteq V(P_m) - \{x_2\}$ and so $|N(u) \cup N(v)| + d(x_2) \leq |V(H_1) \setminus \{u\}| + |V(H_2) \setminus \{v\}| + |\{x_1, x_m\}| + |V(P_m) \setminus \{x_2\}| = |V(H_1)| + |V(H_2)| + |V(P_m)| - 1 \leq n-1$, contrary to (1).

(iii) Let H be a component of $G-P_m$ such that $u, v \in V(H)$ but $uv \notin E(H)$. Since $N_{P_m}(G-P_m) = \{x_1, x_m\}$, then $ux_2 \notin E(G)$ and $vx_2 \notin E(G)$ and $N(u) \cup N(v) \subseteq (V(H) - \{u, v\}) \cup \{x_1, x_m\}$. Thus, $|N(u) \cup N(v)| + d(x_2) \leq |V(H) \setminus \{u, v\}| + |\{x_1, x_m\}| + |V(P_m) \setminus \{x_2\}| \leq n-1$, contrary to (1).

(iv) The statement follows from (ii) and (iii). □

Lemma 2.3. *Let H be a component of $G-P_m$ such that $N_{P_m}(H) = \{x_1, x_i, x_m\}$ and $u \in V(H)$. Then each of the following holds:*

(i) *If there are $x_p, x_q \in V(P_m) \setminus N_{P_m}(H)$ such that $x_px_q \notin E(G)$, then for any vertex $v \in V(G-H) \setminus \{x_p, x_q\}$, either $x_pv \in E(G)$ or $x_qv \in E(G)$.*
(ii) *$G[\{x_2, x_3, \ldots, x_{i-1}\}]$ and $G[\{x_{i+1}, x_{i+2}, \ldots x_{m-1}\}]$ are complete subgraphs.*

(iii) *If* $G - P_m = H = \{u\}$, *then* $G \in \{G_3 \bigvee (K_1 \cup K_h \cup K_t)\}$.

Proof. (i) Let $x_p, x_q \in V(P_m) \setminus N_{P_m}(H)$ such that $x_px_q \notin E(G)$. Then $ux_p \notin E(G)$ and $ux_q \notin E(G)$. Suppose, to the contrary, that there is $v_k \in V(G-H) \setminus \{x_p, x_q\}$ such that $x_px_k \notin E(G)$ and $x_qx_k \notin E(G)$. Then we have $|N(x_p) \cup N(x_q)| + d(u) \leq |V(G)| - |V(H)| - |\{x_p, x_q, x_k\}| + d(u) = |V(G)| - |V(H)| \leq n-1$, contrary to (1).

(ii) To prove that $G[\{x_2, x_3, \ldots, x_{i-1}\}]$ is a complete subgraph, we need to prove the following claims.

<u>Claim 1</u>. $v_2v_k \in E(G)$ for any $i-1 \geq k \geq 4$; $v_{i-1}v_l \in E(G)$ for any $3 \geq l \geq i-3$.

We prove that $v_2v_k \in E(G)$ for any $i-1 \geq k \geq 4$ by induction on $(i-1)-k$. First, we prove $x_2x_{i-1} \in E(G)$, that is, the case when $(i-1)-k = 0$. Suppose, to the contrary, $x_2x_{i-1} \notin E(G)$. Since $x_{i+1} \in V(P_m) \setminus \{x_2, x_{i-1}\}$, then by (i), either $x_{i+1}x_2 \in E(G)$ or $x_{i+1}x_{i-1} \in E(G)$. By Lemma 2.1(ii), $x_{i+1}x_2 \notin E(G)$ and so $x_{i+1}x_{i-1} \in E(G)$. Similarly, we must have $x_{m-1}x_2 \in E(G)$. Since every vertex in $\{x_{i+2}, x_{i+3}, \ldots, x_{m-1}\}$ must be adjacent to either x_2 or x_{i-1}, then there exist two vertices $x_h, x_{h+1} \in \{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$ such that x_h, x_{h+1} are adjacent to x_2, x_{i-1} (or x_{i-1}, x_2), respectively. It follows that G has a longer (x_1, x_m)-path $x_1u[x_i, x_{t-1}][x_2, x_{i-1}][x_t, x_m]$ (or $x_1u[x_i, x_{t-1}][x_{i-1}, x_2][x_t, x_m]$), contrary to (2). This shows that $x_2x_{i-1} \in E(G)$. Now suppose that $x_2x_k \in E(G)$ for any $k \geq s > 4$. We need to prove that $x_2x_{s-1} \in E(G)$. Suppose, to the contrary that $x_2x_{s-1} \notin E(G)$. Since $x_{i+1} \in V(P_m) \setminus \{x_2, x_{s-1}\}$, by (i), either $x_{i+1}x_2 \in E(G)$ or $x_{i+1}x_{s-1} \in E(G)$. By Lemma 2.1(ii), $x_2x_{i+1} \notin E(G)$ and so $x_{i+1}x_{s-1} \in E(G)$. Thus, G has a longer (x_1, x_m)-path $x_1u[x_i, x_s][x_2, x_{s-1}][x_{i+1}, x_m]$, contrary to (2). Hence, $x_2x_{s-1} \in E(G)$ and so $v_2v_k \in E(G)$ for any $i-1 \geq k \geq 4$ by induction. Similarly, we can inductively prove that $v_{i-1}v_l \in E(G)$ for any $3 \leq l \leq i-3$.

<u>Claim 2</u>. $x_px_q \in E(G)$ for any $2 \leq p < q \leq i-1$.

By Claim 1, $v_2v_k \in E(G)$ for any $i-1 \geq k \geq 4$ and $v_{i-1}v_l \in E(G)$ for any $3 \geq l \geq i-3$.

Now suppose that for any $2 \leq p < p'$ and $i-1 \geq q > q'$, where $p < p' < q' < q$, we have $x_px_k \in E(G)$ for any $2 \leq k \leq i-1$ and $x_qx_l \in E(G)$ for any $2 \leq l \leq i-1$. We want to prove that $x_{p'}x_{q'} \in E(G)$. Suppose, to the contrary, that $x_{p'}x_{q'} \notin E(G)$. Since $x_{i+1} \in V(P_m) \setminus \{x_{p'}, x_{q'}\}$, by (i), either $x_{i+1}x_{p'} \in E(G)$ or $x_{i+1}x_{q'} \in E(G)$. If $x_{i+1}x_{p'} \in E(G)$, then G has a longer (x_1, x_m)-path $x_1u[x_i, x_{p'+1}][x_2, x_{p'}][x_{i+1}, x_m]$ and if $x_{i+1}x_{q'} \in E(G)$, then G has a longer (x_1, x_m)-path $x_1u[x_i, x_{q'+1}][x_2, x_{q'}][x_{i+1}, x_m]$, contrary to (2) in either case. Hence, $x_{p'}x_{q'} \in E(G)$ and so $x_px_q \in E(G)$ for any $2 \le p < q \le i-1$ by induction.

By Claim 2, $G[\{x_2, x_3, \ldots, x_{i-1}\}]$ is a complete subgraph.

Similarly, $G[\{x_{i+1}, x_{i+2}, \ldots x_{m-1}\}]$ is also a complete subgraph.

(iii) To prove (iii), we consider the following cases.

Case 1. There exists a vertex $x_t \in \{x_2, x_3, \ldots, x_{i-1}\}$ adjacent to some vertex $x_h \in \{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$.

Let $L = \min\{|\{x_2, x_3, \ldots, x_{i-1}\}|, |\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}|\}$. First suppose that $L = 1$. Without loss of generality, let $|\{x_2, x_3, \ldots, x_{i-1}\}| = 1$, that is $i = 3$. If $x_h \ne x_{m-1}$, then G has a Hamilton (x_1, x_m)-path $x_1ux_3x_2[x_h, x_4][x_{h+1}, x_m]$, contrary to (2). Thus, $x_h = x_{m-1}$. Since $x_1, x_3 \in N_{P_m}(u)$, then by Lemma 2.1(ii), we have $x_2x_4 \notin E(G)$ and so $x_{m-1} \ne x_4$. Since $x_2x_4 \notin E(G)$, then by (i), either $x_2x_m \in E(G)$ or $x_4x_m \in E(G)$. If $x_2x_m \in E(G)$, then G has a Hamilton (x_1, x_m)-path $x_1u[x_3, x_{m-1}]x_2x_m$ and if $x_4x_m \in E(G)$, then G has a Hamilton (x_1, x_m)-path $x_1ux_3x_2[x_{m-1}, x_4]x_m$, contrary to (2) in either case.

Hence, we must have $L \ge 2$. If $x_t \notin \{x_2, x_{i-1}\}$ or $x_h \notin \{x_{i+1}, x_{m-1}\}$, then by the facts that $G[\{x_2, x_3, \ldots, x_{i-1}\}]$ and $G[\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}]$ are complete subgraphs, G has a Hamilton (x_1, x_m)-path $x_1u[x_i, x_{t+1}][x_{t-1}, x_2]x_t[x_h, x_{i+1}][x_{h+1}, x_m]$, contrary to (2). Now let $x_t \in \{x_2, x_{i-1}\}$ and $x_h \in \{x_{i+1}, x_{m-1}\}$. Since $x_2, x_{i+1} \in N^+_{P_m}(u)$ and x_{i-1}, $x_{m-1} \in N^-_{P_m}(u)$, then by Lemma 2.1(ii), $x_2x_{i+1} \notin E(G)$ and $x_{i-1}x_{m-1} \notin E(G)$. Then either $x_{i-1}x_{i+1} \in E(G)$ or $x_2x_{m-1} \in E(G)$. First assume that $x_{i-1}x_{i+1} \in E(G)$. If $x_{i-2}x_{i+2} \notin E(G)$, then by (i), either $x_ix_{i-2} \in E(G)$, whence $x_1ux_ix_{i-2}[x_{i-3}, x_2]x_{i-1}x_{i+1}[x_{i+2}, x_m]$ is a Hamilton (x_1, x_m)-path or $x_ix_{i+2} \in E(G)$, whence $[x_1, x_{i-1}]x_{i+1}[x_{i+3}, x_{m-1}]x_{i+2}x_iux_m$ is a Hamilton (x_1, x_m)-path, contrary to (2) in either case. If $x_{i-2}x_{i+2} \in E(G)$, then $x_2 = x_{i-2}$ and $x_{i+2} = x_{m-1}$ and so $i = 4, m = 7$. Then G has a Hamilton (x_1, x_m)-path $x_1x_2x_6x_5x_3x_4ux_7$, contrary to (2).

Now assume that $x_2x_{m-1} \in E(G)$. If $x_3x_{m-2} \in E(G)$, then $3 = i-1$ and $m-2 = i+1$, that is $i = 4$ and $m = 7$. Then G has a Hamilton (x_1, x_m)-path $x_1ux_4x_5x_3x_2x_6x_7$, contrary to (2). If $x_3x_{m-2} \notin E(G)$, by (i), either $x_3x_m \in E(G)$, whence G has a Hamilton (x_1, x_m)-path $x_1u[x_i, x_{m-1}x_2[x_4, x_{i-1}]x_3x_m$ or $x_{m-2}x_m \in E(G)$, whence G has a Hamilton (x_1, x_m)-path $x_1u[x_i, x_2]x_{m-1}[x_{m-3}, x_{i+1}]x_{m-2}x_m$, contrary to (2) in either case.

Case 2. There is no vertex in $\{x_2, x_3, \ldots, x_{i-1}\}$ adjacent to a vertex in $\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$.

Since $N_{P_m}(u) = \{x_1, x_i, x_m\}$, then $ux_h \notin E(G)$ and by Lemma 2.1(i),

$x_2u \notin E(G)$. By the assumption of Case 2, $x_2x_h \notin E(G)$ and $N(x_2) \cup N(u) \subseteq \{x_1, x_3, x_4, \ldots, x_i, x_m\}$ and for any $x_h \in \{x_{i+1}, x_{i+2}, \ldots x_{m-1}\}$, $N(x_h)\{x_1, x_i, x_{i+1}, \ldots, x_{h-1}, x_{h+1}, x_{m-1}, x_m\}$. Then by (1), we have

$$\begin{aligned} n &\leq |N(x_2) \cup N(u)| + d(x_h) \\ &\leq |\{x_1, x_3, \ldots, x_i, x_m\}| + |\{x_1, x_i, x_{i+1}, \ldots, x_{h-1}, x_{h+1}, x_{m-1}x_m\}| \\ &\leq n. \end{aligned}$$

Thus, x_h must be adjacent to every vertex in $N_{P_m}(u)$. Since x_h is arbitrary, every vertex in $\{x_{i+1}, x_{i+2}, \ldots, x_m\}$ must be adjacent to every vertex in $N_{P_m}(u) = \{x_1, x_i, x_m\}$. Similarly, every vertex in $\{x_2, x_3, \ldots, x_{i-1}\}$ must be adjacent to every vertex in $N_{P_m}(u) = \{x_1, x_i, x_m\}$. This implies $G \in \{G_3 \bigvee (K_1 \cup K_h \cup K_t)\}$. □

Lemma 2.4. *Suppose that $V(G - P_m) = \{u\}$, $d(u) \geq 4$, and $\{x_1, x_m\} \subseteq N_G(u)$. Then $G \in \{G_{n/2} \bigvee K^c_{n/2}\}$.*

Proof. Without loss of generality, let $N_G(u) = \{x_1, x_i, x_j, \ldots, x_r, x_m\}$, where $1 < i < j \leq r < m$. Then $j = r$ if $d(u) = 4$.

<u>Case 1</u>. $x_2x_{m-1} \in E(G)$.

Since $x_{m-2} \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < i < j < m$, then by Lemma 2.1(v), either $x_{i-1}x_{m-2} \in E(G)$ or $x_{j-1}x_{m-2} \in E(G)$. Without loss of generality, suppose $x_{i-1}x_{m-2} \in E(G)$. Then $x_1u[x_i, x_{m-2}][x_{i-1}, x_2]x_{m-1}x_m$ is a Hamilton (x_1, x_m)-path, a contradiction.

<u>Case 2</u>. $x_2x_{m-1} \notin E(G)$.

Then we consider two subcases: $x_{r+1} \neq x_{m-1}$ and $x_{r+1} = x_{m-1}$.

<u>Subcase 2.1</u>. $x_{r+1} \neq x_{m-1}$.

Since $x_{m-1} \in V(P_m) \setminus N^+_{P_m}(u)$ and $1 < i < m$, then by Lemma 2.1(v), either $x_2x_{m-1} \in E(G)$ or $x_{i+1}x_{m-1} \in E(G)$. By the assumption of case 2, $x_2x_{m-1} \notin E(G)$ and so we must have $x_{i+1}x_{m-1} \in E(G)$. Since $x_{r+1} \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < i < j < m$, by Lemma 2.1(v), $x_{r+1}x_{i-1} \in E(G)$ or $x_{r+1}x_{j-1} \in E(G)$ (if $d(u) = 4$, then $j = r$). Then we consider the following two subcases.

<u>Subcase 2.1.1</u>. $x_{r+1}x_{i-1} \in E(G)$.

Since $x_i \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < j < m$, then by Lemma 2.1(v), either $x_ix_{j-1} \in E(G)$, whence G has a Hamilton (x_1, x_m)-path $[x_1, x_i][x_{j-1}, x_{i+1}]x_{m-1}[x_{i-2}, x_j]ux_m$ or $x_ix_{m-1} \in E(G)$, whence G has a Hamilton (x_1, x_m)-path $[x_1, x_{i-1}][x_{r+1}, x_{m-1}]$ $[x_i, x_r]ux_m$, contrary to (2) in either case.

<u>Subcase 2.1.2</u>. $x_{r+1}x_{j-1} \in E(G)$.

Since $x_{r+2} \in V(P_m) \setminus N^+_{P_m}(u)$ and $1 < i < m$, by Lemma 2.1(v), either $x_{r+2}x_2 \in E(G)$, whence by the fact that $x_{r+1}x_{j-1} \in E(G)$, G has a Hamilton (x_1, x_m)-path $x_1u[x_j, x_{r+1}][x_{j-1}, x_2]$ $[x_{r+2}, x_m]$, or $x_{r+2}x_{i+1} \in E(G)$, whence G has a Hamilton (x_1, x_m)-path $[x_1, x_i]u[x_j, x_{r+1}][x_{j-1}, x_{i+1}][x_{r+2}, x_m]$, contrary to (2) in either case.

<u>Subcase 2.2</u>. $x_{r+1} = x_{m-1}$.

Note that both $x_{r+1} = x_{m-1} \in N^+_{P_m}(u)$ and $x_{r+1} = x_{m-1} \in N^-_{P_m}(u)$. Let $x_i, x_j \in N_{P_m}(u)$ be such that $N_{P_m}(u) \cap \{x_{i+1}, x_{i+2}, \dots, x_{j-1}\}) = \emptyset$, then we claim that $x_{i+1} = x_{j-1}$.

Otherwise, since $x_{i+1} \in V(P_m) \setminus N^-_{P_m}(u)$ and $1 < i < m$, then by Lemma 2.1(v), $x_{i-1}x_{i+1} \in E(G)$ or $x_{m-1}x_{i+1} \in E(G)$. Since $x_{r+1} = x_{m-1}$, then $x_{i+1}x_{m-1} \notin E(G)$ and so $x_{i+1}x_{i-1} \in E(G)$. Since $x_{i+2} \in V(P_m) \setminus N^+_{P_m}(u)$ and $1 < i < r < m$, then by Lemma 2.1(v), $x_{i+2}x_2 \in E(G)$, whence G has a Hamilton (x_1, x_m)-path $x_1ux_ix_{i+1}[x_{i-1}, x_2][x_{i+2}, x_m]$, or $x_{i+2}x_{m-1} \in E(G)(x_{i+2}x_{r+1} \in E(G))$, whence G has a Hamilton (x_1, x_m)-path $[x_1, x_{i-1}]x_{i+1}x_iu[x_r, x_{i+2}]x_{r+1}x_m$, contrary to (2) in either case. Therefore, $N_{P_m}(u) = \{x_1, x_3, x_5, x_7, \dots, x_{n-1}\}$. Since P_m is a longest (x_1, x_m)-path, then $\{u, x_2, x_4, x_6, \dots, x_{n-2}\}$ is an independent set. Since for any $x_p, x_q \in \{x_2, x_4, x_6, \dots, x_{n-2}\}$, we have $n \leq |N(x_p) \cup N(x_q)| + d(u) \leq |\{x_1, x_3, x_5, x_7, \dots,\ x_{n-1}\}| + d(u) = n$, then every vertex in $\{x_2,\ x_4,\ x_6,\ \dots,\ x_{n-2}\}$ must be adjacent to every vertex in $\{x_1, x_3, x_5, x_7, \dots, x_{n-1}\}$. Thus, we can get $G \in \{G_{n/2} \bigvee K^c_{n/2}\}$. □

Lemma 2.5. *Suppose that for any* $u \in V(G - P_m)$*, both* $\{x_1, x_m\} \subseteq N_{P_m}(u)$ *and* $N_{P_m}(G - P_m) \neq \{x_1, x_m\}$*. If there exists a component* H *of* $G - P_m$ *such that* $|V(H)| \geq 2$*, then* $G \in \{G_3 \bigvee (K_s \cup K_h \cup K_t)\}$.

Proof. Without loss of generality, let $N_{P_m}(H) = \{x_1, x_i, x_j, \dots, x_r, x_m\}$.

<u>Claim 1</u>. $|N_{P_m}(H)| = 3$.

Otherwise, since G is a 2-connected graph, then $|N_{P_m}(H)| = 2$ or $|N_{P_m}(H)| \geq 4$. If $|N_{P_m}(H)| = 2$, then $N_{P_m}(H) = \{x_1, x_m\}$. By Lemma 2.2(i), $G[\{x_2, x_3, \dots, x_{m-1}\}]$ is a complete subgraph. Since $N_{P_m}(G - P_m) \neq$

$\{x_1, x_m\}$ and G is 2-connected, then $G - P_m$ has a component S such that $x_i \in N_{P_m}(S) \setminus \{x_1, x_m\}$ and $x_j \in N_{P_m}(S)$. Without loss of generality, suppose that $1 < i < j \leq m$. Since $x_i, x_j \in N_{P_m}(H)$, there exist $x_i', x_j' \in V(H)$ such that $x_i x_i', x_j x_j' \in E(G)$. Let P' denote an (x_i', x_j')-path in H. Hence, G has a longer (x_1, x_m)-path $[x_1, x_{i-1}][x_{i+1}, x_{j-1}]x_i P'[x_j, x_m]$, contrary to (2). Now suppose $|N_{P_m}(H)| \geq 4$ and $u \in V(H)$. Let $v \in V(H) \setminus \{u\}$. By Lemma 2.1(v), $vx_2 \in E(G)$ or $vx_{i+1} \in E(G)$. Since $x_1 \in N_{P_m}(v)$, then by Lemma 2.1(i), $x_2 \notin N_{P_m}(v)$ and so $x_{i+1}v \in E(G)$. Since $|N_{P_m}(H)| \geq 4$, then there is $x_j \in N_{P_m}(H) \setminus \{x_1, x_i, x_m\}$. By the same argument, we have $x_{j+1}v \in E(G)$ and so $[x_1, x_i]u[x_j, x_{i+1}]v[x_{j+1}, x_m]$ is a longer (x_1, x_m)-path, contrary to (2).

Let $N_{P_m}(H) = \{x_1, x_i, x_m\}$. By Lemma 2.3(ii), we have the following Claim 2.

<u>Claim 2</u>. $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ and $G[\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}]$ are all complete subgraphs.

Since G is 2-connected and $|V(H)| \geq 2$, then there are $x_1', x_i' \in V(H)$ such that $x_1' \neq x_i'$ and $x_1 x_1', x_i x_i' \in E(G)$ or there are $x_i'', x_m'' \in V(H)$ such that $x_i'' \neq x_m''$ and $x_i x_i'', x_m x_m'' \in E(G)$. Without loss of generality, suppose there are $x_1', x_i' \in V(H)$ such that $x_1' \neq x_i'$ and $x_1 x_1', x_i x_i' \in E(G)$. Let P' denote an (x_1', x_i')-path in H.

<u>Claim 3</u>. $G - P_m$ is a connected subgraph.

Otherwise, let S be another component of $G - P_m$. By Lemma 2.3(i), every vertex in S must be adjacent to one of x_2 and x_{i+1}. Since every vertex in S is adjacent to x_1, by Lemma 2.1(i), no vertex in S can be adjacent to x_2 and so every vertex in S must be adjacent to x_{i+1}. If $x_2 x_{i+2} \in E(G)$, then we can get a longer (x_1, x_m)-path $x_1 P'[x_i, x_2][x_{i+2}, x_m]$, contrary to (2). Then we have $x_2 x_{i+2} \notin E(G)$. By Lemma 2.3(i) and Lemma 2.1(i) again, every vertex in S must be adjacent to x_{i+2}, contradicting Lemma 2.1(i).

<u>Claim 4</u>. H is a complete subgraph.

Otherwise, let $u, v \in V(H)$ such that $uv \notin E(G)$. Then we have $|N(x_2) \cup N(x_{i+1})| + d(u) \leq |V(P_m)| + |V(H)| - |\{x_2, x_{i+1}, u, v\}| + |N_{P_m}(H)| \leq n - 1$, contrary to (1).

<u>Claim 5</u>. For any $u \in V(H)$, u must be adjacent to every vertex of $N_{P_m}(H)$.

Otherwise, there exists $u \in V(H)$ such that $ux_i \notin E(G)$. Then $|N(x_2) \cup N(x_{i+1})| + d(u) \leq |V(P_m) \setminus \{x_2, x_{i+1}\}| + |V(H) \setminus \{u\}| + |N_{P_m}(H) \setminus \{x_i\}| \leq n-1$, contrary to (1). Similarly, for every vertex u in $\{x_2, x_3, \ldots, x_{i-1}\}$ or $\{x_{i+1}, x_{i+2}, \ldots, x_{m-1}\}$, u must be adjacent to every vertex in $N_{P_m}(H) = \{x_1, x_i, x_m\}$. Then by Claim 1, Claim 2, Claim 3, Claim 4 and Claim 5, we have $G \in \{G_3 \bigvee (K_s \cup K_h \cup K_t)\}$. □

Proof of Theorem. Let G be a 2-connected graph such that (1) holds. Suppose that G is not Hamiltonian-connected and so we may assume that there exist $x, y \in V(G)$ such that G has no Hamilton (x, y)-path and such that (2) holds. We want to show that $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \bigvee K^c_{n/2}, G_2 : (K_s \cup K_h \cup K_t), G_3 \bigvee (K_s \cup K_h \cup K_t)\}$. We consider the following cases.

Case 1. There exists a vertex u in $G - P_m$ such that ux_1 or $ux_m \notin E(G)$.

Without loss of generality, suppose $ux_m \notin E(G)$. Let G^* be the component of $G - P_m$ containing u. Since G is 2-connected, then $|N_{P_m}(G^*)| \geq 2$.

Subcase 1.1. $|N_{P_m}(G^*)| \geq 3$.

In this case, there exist two distinct vertices $x_{i+1}, x_{j+1} \in N^+P_m(G^*)$ such that $x_{i+1}x_{j+1} \notin E(G)$. Then we have the following claim.

Claim. For any vertex $v \in N_{G-P_m}(u) \cup N^+_{P_m}(u)$, $vx_{i+1} \notin E(G)$ and $vx_{j+1} \notin E(G)$.

By Lemma 2.1(ii), for any vertex $v \in N^+P_m(u)$, $vx_{i+1} \notin E(G)$ and $vx_{j+1} \notin E(G)$. Now suppose there is $v \in N_{G-P_m}(u)$ such that $vx_{i+1} \in E(G)$ or $vx_{j+1} \in E(G)$. Without loss of generality, suppose that $vx_{i+1} \in E(G)$. Since $x_i \in N_{P_m}(G^*)$, there exists $x'_i \in V(G^*)$ such that $x_i x'_i \in E(G)$. Let P' denote an (x'_i, v)-path in G^*. Then we get a longer (x_1, x_m)-path $[x_1, x_i]P_1[x_{i+1}, x_m]$, contrary to (2).

Since $x_{i+1}, x_{j+1} \in N^+P_m(G^*)$, by Lemma 2.1(i), $ux_{i+1} \notin E(G)$ and $ux_{j+1} \notin E(G)$. By the above Claim, we have $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N_{G-P_m}(u) \cup N^+_{P_m}(u)| - |\{u\}|$. Since $|N^+_{P_m}(u)| = |N_{P_m}(u)|$, then $|N_{G-P_m}(u) \cup N^+_{P_m}(u)| = |N_{G-P_m}(u) \cup N_{P_m}(u)| = |N(u)|$ and so $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N(u)| - |\{u\}| = n - |N(u)| - 1$, which implies $|N(x_{i+1}) \cup N(x_{j+1})| + d(u) \leq n - 1$, contrary to (1).

Subcase 1.2. $|N_{P_m}(G^*)| = 2$.

If $N_{P_m}(G^*) \neq \{x_1, x_m\}$, then by the argument similar to that in above Subcase 1.1, we can obtain a contradiction. Then we have $N_{P_m}(G^*) = \{x_1, x_m\}$. By Lemma 2.2(i), $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is a complete subgraph.

If there exists a vertex $x_i \in V(P_m) \setminus \{x_1, x_m\}$ satisfying the condition x_i is adjacent to some vertex of $G - P_m$, then there exists a component H of $G - P_m - G^*$ such that x_i is adjacent to some vertex of H. Since G is 2-connected, then there exist $x_{i+1}, x_{j+1} \in N^+_{P_m}(H)$ or $x_{i-1}, x_{j-1} \in N^-_{P_m}(H)$. Since $G[\{x_2, x_3, \ldots, x_{m-1}\}]$ is a complete subgraph, then $x_{i+1}x_{j+1}$ and $x_{i-1}x_{j-1} \in E(G)$, contrary to Lemma 2.1(ii). Then we have $N_{P_m}(G - P_m) = \{x_1, x_m\}$. By Lemma 2.2(iv), we have $G \in \{G_2 : (K_s \cup K_h), G_2 : (K_s \cup K_h \cup K_t)\}$.

Case 2. For any vertex u in $G - P_m$, u is adjacent to x_1 and x_m.

If $N_{P_m}(G - P_m) = \{x_1, x_m\}$, by Lemma 2.2(iv), we have $G \in \{G_2 : (K_s \cup K_h), G_2 : (K_s \cup K_h \cup K_t)\}$. In the following, we suppose that $N_{P_m}(G - P_m) \neq \{x_1, x_m\}$. Then there exists a component G^* of $G - P_m$ such that $N_{P_m}(G^*) \cap (V(P_m) \setminus \{x_1, x_m\}) \neq \emptyset$.

Subcase 2.1. $|V(G - P_m)| = |\{u\}| = 1$.

Since u is adjacent to x_1 and x_m and $N_{P_m}(u) \cap (V(P_m) \setminus \{x_1, x_m\}) \neq \emptyset$, then $d(u) \geq 3$. If $d(u) = 3$, then by Lemma 2.3(iii), $G \in \{G_3 \bigvee (K_1 \cup K_h \cup K_t)\}$. If $d(u) \geq 4$, then by Lemma 2.4, $G \in \{G_{n/2} \bigvee K^c_{n/2}\}$.

Subcase 2.2. $|V(G - P_m)| \geq 2$.

If there exists a component H of $G - P_m$ such that $|V(H)| \geq 2$, then by Lemma 2.5, $G \in \{G_3 \bigvee (K_s \cup K_h \cup K_t)\}$. Now we suppose that for every component H of $G - P_m$, $|V(H)| = 1$.

Claim. For any vertex $u \in V(G - P_m)$, $N_{P_m}(u) \leq 3$.

Otherwise, let $N_{P_m}(u) \geq 4$ and $N_{P_m}(u) = \{x_1, x_i, x_j, \ldots, x_m\}$ with $1 < i < j < m$. Since $|V(G - P_m)| \geq 2$, there exists a vertex $v \in V(G - P_m) \setminus \{u\}$. By Lemma 2.1(v), $vx_2 \in E(G)$ or $vx_{i+1} \in E(G)$. Since $x_1 \in N_{P_m}(v)$, then by Lemma 2.1(i), $vx_2 \notin E(G)$ and so $vx_{i+1} \in E(G)$. Similarly, $vx_{j+1} \in E(G)$, contrary to Lemma 2.1(iv).

Since $N_{P_m}(G^*) \cap (V(P_m) \setminus \{x_1, x_m\}) \neq \emptyset$, then there exists $v \in V(G - P_m)$ such that $|N_{P_m}(v)| = 3$. Without loss of generality, let $N_{P_m}(v) = \{x_1, x_i, x_m\}$. Let $w \in V(G - P_m) \setminus \{v\}$. By Lemma 2.1(v), either $wx_2 \in E(G)$ or $wx_{i+1} \in E(G)$. Since $x_1 \in N_{P_m}(w)$, then $wx_2 \notin E(G)$ and so $wx_{i+1} \in E(G)$. Similarly, $wx_{i-1} \in E(G)$. Then $x_{i-1}, x_{i+1}, x_1, x_m \in N_{P_m}(w)$, namely, $|N_{P_m}(w)| \geq 4$, contrary to the claim that for any vertex $u \in V(G - P_m)$, $N_{P_m}(u) \leq 3$. □

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