A^p IS NOT AN ALGEBRA FOR $1 < p < 2$

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Abstract. Let A^p be the Banach space of all continuous functions on the torus $\mathbb{T} = \{z \in \mathbb{C} | |z| = 1\}$ whose Fourier coefficients are in ℓ^p. We show that A^p is not an algebra for all $1 < p < 2$.

1. Introduction

A compact subset $K \subset \mathbb{T}$ is said to be Helson if K is a closed subset of the torus on which every continuous function has a sequence of Fourier coefficients in ℓ^1. A Carleson set is a Helson set for which $\widehat{f}(n) = 0$ for all $n < 0$ for all continuous functions f. In 1960 Wik [9] proved the more interesting relationship that all Helson sets are Carleson sets. Later in 1968 A. Bernard [1] gave a much more concise proof of this fact. His proof was an application of the following theorem.

Theorem 1.1. *(Bernard [1]) Let A be a complex Banach function algebra on K. If $\mathrm{Re}A = \{\mathrm{Re}f | f \in A\}$ is uniformly closed then $A = C(K)$.*

Bernard's proof is a combination of this theorem and the fact that A^1, the space of all continuous functions on the torus $\mathbb{T}$ with summable Fourier coefficients, has the property that $\mathrm{Re}A^1 = \mathrm{Re}A^{1+}$, where $A^{1+} = \{f \in A^1 | \widehat{f}(n) = 0 \quad \text{for all } n < 0\}$. In 1972, M. Gregory [2] introduced the notion of p-Helson and p-Carleson sets. For these variations on Helson and Carleson sets the Fourier coefficients need to be in ℓ^p rather than ℓ^1. Hence a 1-Helson set is the original Helson set. Gregory [2] asked the natural question: "is every p-Helson set a p-Carleson set?" He noted that since every closed subset of the torus is a 2-Helson set and all 2-Carleson sets have Lebesgue measure zero, the answer is no for $p = 2$. But what of the case $1 < p < 2$?

We define A^p to be the space of all continuous functions on $\mathbb{T}$ with pth power summable Fourier coefficients again under pointwise multiplication. For $p \geq 1$, A^p is a Banach function space under the norm $\|\cdot\| = \max\{\|\cdot\|_\infty, \|\widehat{\cdot}\|_{\ell^p}\}$ (A^p is not complete in either norm alone if $1 < p < 2$). The Banach spaces studied here are different than the Banach algebras $A_p(G)$ studied by Larsen, Liu, and Wang [3] and Warner [8]. Their spaces of functions are all integrable functions on a locally compact abelian group G

whose Fourier transforms are in $L^p(\widehat{G})$. Also, the multiplication in $A_p(G)$ is convolution rather than pointwise multiplication.

Can we apply Bernard's technique to determine if p-Helson and p-Carleson sets are the same for $1 < p < 2$? If so the only fact to be checked would be whether or not $\mathrm{Re}A^p = \mathrm{Re}A^{p+}$. However, as this paper will procede to show, we cannot use Bernard's technique since A^p is not an algebra on the range $1 < p < 2$. It has been shown [4] that A^p is not a Banach algebra for any $1 < p < p_0$, for a certain $p_0, 1 < p_0 < 2$. In this paper we extend that result to the entire range $1 < p < 2$ via the following theorem.

Theorem 1.2. *The Banach space, A^p, is not closed under pointwise multiplication for $1 < p < 2$.*

In the definition of A^p the underlying space is $\mathbb{T}$. What if the domain of the functions in A^p was an arbitrary locally compact group G? Define $A^p(G)$ to be the set of all continuous functions on G whose Fourier transforms are in $L^p(\widehat{G})$, where $\widehat{G}$ is the Pontrjagin dual of G and the underlying measure is a left Haar measure on $\widehat{G}$. With the L^p conjecture in mind, Theorem 1.2 and the discussion following leads one to the following conjecture.

Conjecture 1.3. *If G is a locally compact group and $1 < p < 2$ then $A^p(G)$ is closed under pointwise multiplication if and only if $\widehat{G}$ is compact.*

The L^p conjecture states: If $p > 1$ and $\widehat{G}$ is a locally compact group, $L^p(\widehat{G})$ is closed under convolution if and only if $\widehat{G}$ is compact. For $\widehat{G}$ abelian and $p > 1$, the L^p conjecture was first proven by Żelazko and Urbanik [10], [7]. For $p > 2$ and $\widehat{G}$ not necessarily abelian, the conjecture was proven by Rajagopalan [5] and Żelazko [11], and a self contained proof of the entire L^p conjecture was given by Saeki [6]. Conjecture 1.3 is slightly stronger, as the L^p conjecture is an immediate corollary in the range $1 < p < 2$.

Throughout the remainder of the paper $A^p = A^p(\mathbb{T})$.

As in the original paper we will use a well-known fact from Banach function space theory. For completeness, we give the proof here.

Theorem 1.4. *Let A be a Banach function space. If A is closed under multiplication then there exists a constant M such that for all $u, v \in A$, $\|uv\| \le M\|u\|\|v\|$.*

Proof. For each $v \in A$ define the linear operator $\Lambda_v \colon A \to A$ by $\Lambda_v u = uv$. By the closed graph theorem Λ_v is continuous. Define the linear operator $\Gamma \colon A \to \mathcal{B}(A)$ by $\Gamma(v) = \Lambda_v$. Another application of the closed graph theorem implies that Γ is continuous. Thus,

$$\|uv\| = \|\Lambda_v u\| \leq \|\Lambda_v\| \|u\| = \|\Gamma(v)\| \|u\| \leq \|\Gamma\| \|v\| \|u\|.$$

□

In order to obtain an example to illustrate that A^p is not closed under pointwise multiplication we need a family of polynomials g_n such that

$$\lim_{n\to\infty} \frac{\|g_n^2\|}{\|g_n\|^2} = \infty. \tag{1.1}$$

If $\|\widehat{g_n^2}\|_p \leq \|g_n^2\|_\infty$ or if $\|\widehat{g_n^2}\|_p \leq \|\widehat{g_n}\|_\infty^2$ then $\|g_n^2\| \leq \|g_n\|^2$. So we are looking for polynomials g_n with relatively small sup norms such that g_n^2 has coefficients with relatively large pth power sums. A natural approach then is to use Riesz products to ensure no cancelation among positive and negative terms when squaring. This is precisely what was done in the earlier work. In this example we use a variation of a Reisz product to obtain the full result.

2. Generalized Reisz Product

We begin with a result which allows us to extend results about degree 3 polynomials to polynomials of arbitrarily high degree.

Theorem 2.1. *Let $f(z)$ be a third degree polynomial on the torus $\mathbb{T}$ with real coefficients, and let*

$$g_n(z) = \prod_{j=0}^{n} f(z^{7^j}).$$

Then

$$\|g_n\|_\infty \leq \|f\|_\infty^n, \tag{2.1}$$

$$\|\widehat{g_n}\|_p = \|\widehat{f}\|_p^n, \tag{2.2}$$

and

$$\|\widehat{g_n^2}\|_p = \|\widehat{f^2}\|_p^n. \tag{2.3}$$

Proof. Inequality (2.1) is a direct consequence of a well-known fact about sup norms.

Let $f(z) = a + bz + cz^2 + dz^3$. Write $g_n(z) = g_{n-1}(z) f(z^{7^n})$, and notice that $g_{n-1}(z)$ is a polynomial of degree at most $\frac{7^n-1}{2}$. Therefore, the terms $ag_{n-1}(z)$, $bz^{7^n} g_{n-1}(z), cz^{2\cdot 7^n} g_{n-1}(z)$, and $dz^{3\cdot 7^n} g_{n-1}(z)$ have no common powers of z. So

$$\begin{aligned}
&\|g_{n-1}\widehat{(z)f}(z^{7^n})\|_p^p \\
&= \|a\widehat{g_{n-1}(z)}\|_p^p + \|bz^{7^n}\widehat{g_{n-1}}(z)\|_p^p + \|cz^{2\cdot 7^n}\widehat{g_{n-1}}(z)\|_p^p + \|dz^{3\cdot 7^n}\widehat{g_{n-1}}(z)\|_p^p \\
&= \|\widehat{g_{n-1}}\|_p^p(a^p + b^p + c^p + d^p) \\
&= \|\widehat{g_{n-1}}\|_p^p\|\widehat{f}\|_p^p.
\end{aligned}$$

Equation (2.2) then follows by induction and taking pth roots.

Let $f^2(z) = a^2 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6$. Write $g_n^2(z) = g_{n-1}^2(z) f^2(z^{7^n})$, and notice that $g_{n-1}^2(z)$ is a polynomial of degree at most $7^n - 1$. So when $g_{n-1}^2(z)$ and $f^2(z^{7^n})$ are multiplied there will be no common powers of z between the terms $a^2 g_{n-1}(z)$, $a_1 z^{7^n} g_{n-1}(z)$, $a_2 z^{2\cdot 7^n} g_{n-1}(z)$, $a_3 z^{3\cdot 7^n} g_{n-1}(z)$, $a_4 z^{4\cdot 7^n} g_{n-1}(z)$, $a_5 z^{5\cdot 7^n} g_{n-1}(z)$, and $a_6 z^{6\cdot 7^n} g_{n-1}(z)$. So using the same calculations as for (2.2), we see

$$\|g_{n-1}^2\widehat{(z)f^2}(z^{7^n})\|_p^p = \|\widehat{g_{n-1}^2}\|_p^p\|\widehat{f^2}\|_p^p.$$

Equation (2.3) then follows by inductions and taking pth roots. □

3. Choosing Coefficients

With Theorem 2.1 in hand we can now look for a degree 3 polynomial, $f(x)$, such that $\|f\|_\infty$ is small and $\|\widehat{f^2}\|_p$ is large. We are now charged with determining the coefficients of $f(z) = a + bz + cz^2 + dz^3$. Since constants factor out of norms we may as well assign one of the coefficients to be one. Therefore we can write $f(z) = a + bz + cz^2 + z^3$. Next we choose a, b and c to create a relatively small sup norm. Let $z \in \mathbb{T}$, that is $z = e^{i\theta}$ where $\theta \in \mathbb{R}$ thus,

$$|f(z)|^2 = a^2 + b^2 + c^2 + 1 + 2(bc + ab + c)\cos\theta + 2(b + ac)\cos(2\theta) + 2a\cos(3\theta).$$

Also,

$$\|f\|_\infty^2 \le a^2 + b^2 + c^2 + 1 + 2|bc + ab + c| + 2|b + ac| + 2|a|.$$

In order for this to be small we choose a, b and c so that

$$\begin{aligned}
bc + ab + c &= 0 \\
b + ac &= 0.
\end{aligned}$$

The above system of equations is satisfied when

$$\begin{aligned}
b &= a^2 - 1 \\
c &= \frac{1}{a} - a.
\end{aligned}$$

Notice that we could have made the sup norm even smaller if we had insisted also that $a = 0$. However, that would force $f(z)$ to be z^3 and $\|\widehat{f^2}\|_p = 1$

that is not nearly large enough. Our polynomial, $f(z)$, still needs to have a large sum of pth power coefficients when squared, however we still have yet to chose a. We will use that remaining choice to make the ℓ^p norm of the coefficients of $f^2(z)$ large. Let $f_a(z) = a + (a^2-1)z + (\frac{1}{a} - a)z^2 + z^3$. A few calculations give the following results.

$$\begin{aligned} \|f_a\|_\infty^2 &= a^4 + 2a + \frac{1}{a^2}, \\ \|\widehat{f_a}\|_p^{2p} &= \left(\frac{a^p+1}{a^p}\right)^2 \left(a^p + (a^2-1)^p\right)^2, \\ \|\widehat{f_a^2}\|_p^p &= a^{2p} + (2a^3-2a)^p + (a^4-4a^2+3)^p + \left(2a^3 - 6a + \frac{2}{a}\right)^p \\ &\quad + \left(4a^2 - 4 + \frac{1}{a^2}\right)^p + \left(2a - \frac{2}{a}\right)^p + 1. \end{aligned}$$

We can use the generalized binomial theorem to give asymptotic descriptions of these three norms.

$$\|f_a\|_\infty^{2p} = a^{4p} + 2pa^{4p-3} + o(a^{4p-3}) \tag{3.1}$$

$$\|\widehat{f_a}\|_p^{2p} = a^{4p} + 4a^{3p} - 2pa^{4p-2} + o(a^{3p}) \tag{3.2}$$

$$\|\widehat{f_a^2}\|_p^p = a^{4p} + 2^{p+1}a^{3p} + o(a^{3p}) \tag{3.3}$$

The next two theorems will show that given $1 < p < 2$ we can choose a large enough so that f_a will have the desired property.

Theorem 3.1. *For all $1 < p < 2$ there exists a constant B_p such that for all $a \geq B_p$ we have $\|\widehat{f_a}\|_p^2 < \|\widehat{f_a^2}\|_p$.*

Proof. For $p > 1$, $2^{p+1} > 4$. Therefore this theorem is a direct consequence of (3.2) and (3.3). □

Theorem 3.2. *For all $1 \leq p < 2$ there exists a constant C_p such that for all $a \geq C_p$ we have $\|f_a\|_\infty < \|\widehat{f_a}\|_p$.*

Proof. Using (3.2) we see for $p < 2$, we have $\|\widehat{f_a}\|_p^{2p} = a^{4p} + 4a^{3p} + o(a^{3p})$. Similarly, using (3.1) we see for $p < 2$, we have $\|f_a\|_\infty^{2p} = a^{4p} + o(a^{3p})$. Therefore, there exists an a large enough so that $\|f_a\|_\infty < \|\widehat{f_a}\|_p$. □

Note that in equation (3.2) we include the term $-2pa^{4p-2}$ to illustrate that this proof will not work for the case $p = 2$.

For any $1 < p < 2$, Let $a > \max\{B_p, C_p\}$ where B_p is obtained from Theorem 3.1 and C_p is obtained from Theorem 3.2. For such a value of a we have

$$\|f_a\|_\infty^2 < \|\widehat{f_a}\|_p^2 < \|\widehat{f_a^2}\|_p.$$

Let $g_n(z) = \prod_{j=0}^{n} f_a(z^{7^j})$. We now verify the necessary limit

$$\lim_{n\to\infty} \frac{\|g_n^2\|}{\|g_n\|^2} = \lim_{n\to\infty} \frac{\|\widehat{g_n^2}\|_p}{\|\widehat{g_n}\|_p^2} = \lim_{n\to\infty} \frac{\|\widehat{f_a^2}\|_p^n}{\|\widehat{f}_a\|_p^{2n}} = \infty.$$

Therefore A^p is not an algebra for $1 < p < 2$, and we have proven Theorem 1.2.

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