No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

175. *Proposed by N. J. Kuenzi, Oshkosh, Wisconsin.*

The positive integer 45 can be written as a sum of five consecutive positive integers (SCPI): $45 = 7 + 8 + 9 + 10 + 11$; furthermore, 45 can be written as a SCPI in *exactly* five ways, namely, $45 = 22 + 23 = 14 + 15 + 16 = 7+8+9+10+11 = 5+6+7+8+9+10 = 1+2+3+4+5+6+7+8+9+10$. Is there a positive integer that can be written as a sum of 2009 consecutive positive integers and which can be written as a SCPI in *exactly* 2009 ways?

Comment and Solution by Calvin A. Curtindolph, Fox Lake, Wisconsin. The solution by Kandasamy Muthuvel to Problem 175 which appeared in the February 2011 issue of the Missouri Journal of Mathematical Sciences is incorrect. 3^{2009} may be expressed as a sum of consecutive positive integers (SCPI) in exactly 2009 ways, but 3^{2009} is not the sum of 2009 consecutive positive integers (hereafter abbreviated S(2009)CPI. SnCPI will abbreviate "sum of n consecutive positive integers"). If 3^{2009} were a S(2009)CPI, the initial term in the sum would be

$$a = \frac{3^{2009}}{2009} - 1004 = \frac{3^{2009}}{7^2 \cdot 41} - 1004,$$

which clearly cannot be an integer.

Further, the assertion that 3^s, where $s > 1$, can be written as a SsCPI and can be written as a SCPI in exactly s ways is false. While the proof does show that 3^s can be written as a SCPI in exactly s ways by showing the cardinality of the set $\{n \ : \ n = 3^t - 1$ for some integer t with $1 \le t \le s/2$ or $n = 2 \cdot 3^t - 1$ for some integer t with $0 \le t < s/2\}$ is s, the proof does not show, indeed, cannot show, that s is an element of the set. It is not. 3^s cannot be expressed as a SsCPI, at least not for arbitrary $s > 1$.

As noted in the journal, I also submitted a solution to this problem. In that solution, I noted that it could be easily generalized. I now submit a generalization of my previous solution useful for constructing a set of integers S expressible as a SnCPI and expressible as a SCPI in exactly n ways (where possible).

I note here that a S(N)CPI is, by definition:

$$S = b + (b+1) + \cdots + (b + (N-1))$$

where b is a positive integer and N is an integer greater than 1. This definition gives rise to the equivalent equations

$$S = \frac{N(2b + N - 1)}{2},$$

$$b = \frac{S}{N} - \frac{N-1}{2},$$

$$\text{and } N^2 + (2b-1)N - 2S = 0.$$

The positive solution for N of this last equation is

$$N = \frac{-(2b-1) + \sqrt{(2b-1)^2 + 8S}}{2}.$$

Recall from elementary number theory that if $d(X)$ denotes the number of positive divisors of the positive integer X, and the prime factorization of $X = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the p_i are distinct primes, then $d(X) = (e_1+1)(e_2+1)\cdots(e_k+1)$.

My main result is Theorem 1.

Theorem 1. *Let n have prime factorization*

$$n = 2^{d_0} q_1^{d_1} \cdots q_j^{d_j},$$

where $d_0 = 0$ or 1, the q_i are distinct odd primes and the d_i are positive integers. Suppose there exist nonnegative integers $e_1, \ldots, e_k$, $k \geq j$, such that

$$(d_1 + e_1 + 1) \cdots (d_j + e_j + 1)(e_{j+1} + 1) \cdots (e_k + 1) = n + 1.$$

Finally, suppose there exist distinct odd primes $p_{j+1}, \dots, p_k$, each unequal to any of the q_i, such that

$$2^{d_0} n < 2q_1^{e_1} \cdots q_j^{e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}.$$

Then the positive integer

$$S = q_1^{d_1+e_1} \cdots q_j^{d_j+e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k} = 2^{-d_0} n q_1^{e_1} \cdots q_j^{e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}$$

is a $SnCPI$ which may also be expressed as a SCPI in exactly n ways.

Proof. Assume the hypotheses of the theorem are satisfied. Consider the positive integers

$$S = q_1^{d_1+e_1} q_2^{d_2+e_2} \cdots q_j^{d_j+e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}$$

$$\text{and } 2S = 2q_1^{d_1+e_1} q_2^{d_2+e_2} \cdots q_j^{d_j+e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}.$$

S is an odd integer with $n+1$ positive divisors. Since $2S$ contains only one factor of 2, $2S$ has $2(n+1)$ positive divisors, $n+1$ of which are less than $\sqrt{2S}$ and $n+1$ of which are greater than $\sqrt{2S}$. Let $F_1, F_2, \dots, F_{n+1}$ be the divisors of $2S$ which are greater than $\sqrt{2S}$, arranging them in any order such that $F_{n+1} = 2S$. For each i, $1 \le i \le n+1$, consider $F_i - 2S/F_i$. Clearly $F_i - 2S/F_i$ is a positive integer. If F_i is odd, then $2S/F_i$ is even and if F_i is even then $2S/F_i$ is odd, hence $F_i - 2S/F_i$ is odd. Set $2b - 1 = F_i - 2S/F_i$, noting that $b = (F_i + 1)/2 - (S/F_i)$ is a positive integer.

Now set

$$N = \frac{-(F_i - 2S/F_i) + \sqrt{(F_i - 2S/F_i)^2 + 8S}}{2}$$

$$= \frac{-(F_i - 2S/F_i) + \sqrt{(F_i + 2S/F_i)^2}}{2} = \frac{2S}{F_i}.$$

The first of the expressions is the positive root of the quadratic equation

$$N^2 + (F_i - 2S/F_i)N - 2S = 0.$$

This gives

$$S = \frac{N^2 + (F_i - 2S/F_i)N}{2} = \frac{N^2 + (2b-1)N}{2} = N(b-1) + \frac{N(N+1)}{2}$$

$$= \sum_{k=1}^{N} b + (k-1) = b + (b+1) + \cdots + (b + (N-1)).$$

We have established S as a S($2S/F_i$)CPI (starting with b) for each i, $1 \leq i \leq n+1$. Since $F_{n+1} = 2S$, the "SCPI" corresponding to the factor F_{n+1} is the "trivial SCPI" containing only one term: $S = S$. Since we do not mean such a trivial sum by the phrase, we are left with exactly n ways of expressing S as a SCPI.

We now need to show that one of the integers $N = 2S/F_i$ is equal to n. Consider

$$F = 2^{1-d_0} q_1^{e_1} \cdots q_j^{e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k},$$

so that

$$2S/F = 2^{d_0} q_1^{d_1} \cdots q_j^{d_j} = n.$$

We need to show that $F > \sqrt{2S}$, or equivalently $2S/F = n < \sqrt{2S}$.

The condition that

$$2^{d_0} n < 2 q_1^{e_1} \cdots q_j^{e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}$$

implies that

$$(2^{-d_0} n)(2^{d_0} n) < (2^{-d_0} n)(2 q_1^{e_1} \cdots q_j^{e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}).$$

Thus, $n^2 < 2S$ and so $n < \sqrt{2S}$. Hence, for one of the factors F_i, $N = 2S/F_i = n$, and S is a SnCPI. $\square$

We now apply Theorem 1 to the original problem. Here, $n = 2009 = 7^2 \cdot 41$, so that $d_0 = 0$, $j = 2$, $q_1 = 7$, $d_1 = 2$, $q_2 = 41$, and $d_2 = 1$. $n + 1 = 2010 = 2 \cdot 3 \cdot 5 \cdot 67$. There are certainly nonnegative integers e_1, e_2, e_3, and e_4 satisfying $(e_1+3)(e_2+2)(e_3+1)(e_4+1) = 2 \cdot 3 \cdot 5 \cdot 67$. Indeed, noting that e_3 and/or e_4 may be zero, we have several choices. We need only restrict our choices of the e_i and of the primes p_3 and p_4 to the condition that $2009 < 2 \cdot 7^{e_1} \cdot 41^{e_2} \cdot p_3^{e_3} \cdot p_4^{e_4}$, or $2 \cdot 7^{e_1-2} \cdot 41^{e_2-1} \cdot p_3^{e_3} \cdot p_4^{e_4} > 1$.

For such choices, $S = 7^{e_1+2} \cdot 41^{e_2+1} \cdot p_3^{e_3} \cdot p_4^{e_4}$ is a S(2009)CPI expressible as an SCPI in exactly 2009 ways.

Relating Theorem 1 to the example given in the problem, that 45 is a S5CPI expressible as a SCPI in exactly 5 ways, we may determine that the complete set of integers so expressible is $\{5p^2 \ : \ p$ an odd prime, $p \neq 5\} \cup \{25p \ : \ p$ odd prime, $p \neq 5\} \cup \{5^5\}$.

Finally, I believe the converse of Theorem 1 to be true. I also propose a nice corollary, assuming the converse of Theorem 1.

Conjecture 2. *Let*

$$n = 2^{d_0} q_1^{d_1} \cdots q_j^{d_j},$$

where $d_0 = 0$ or 1, the q_i are distinct odd primes and the d_i are positive integers. If any of the hypotheses of Theorem 1 fail, that is if

(1) *there are no nonnegatve integers $e_1, \ldots, e_k$, $k \geq j$, such that*

$$(d_1 + e_1 + 1) \cdots (d_j + e + j + 1)(e_{j+1} + 1) \cdots (e_k + 1) = n + 1$$

or

(2) *there are no distinct odd primes $p_{j+1}, \ldots p_k$, each unequal to any of the q_i, such that*

$$2^{d_0} n < 2q_1^{e_1} \cdots q_j^{e_j} p_{j+1}^{e_{j+1}} \cdots p_k^{e_k},$$

then there is no positive integer S which is a SnCPI and which may be expressed as a SCPI in exactly n ways.

Conjecture 3 (dependent on Conjecture 2)**.** *$S = p^n$, where p is an odd prime is a SnCPI and may be expressed as a SCPI in exactly n ways if and only if $n = p^{n-e}$, where $e \geq n/2$ or $n = 2p^{n-e}$, where $e > n/2$.*