SOME INEQUALITIES IN PTOLEMAIC SPACES

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ABSTRACT. In this note, we establish some new interesting inequalities in Ptolemaic spaces.

1. Introduction and Main Results

Ptolemy's inequality in $\mathbb{R}^2$ states: If A, B, C, D are vertices of a quadrilateral, then

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD \tag{1.1}$$

with equality if and only if $ABCD$ is a convex cyclic quadrilateral.

I. J. Schoenberg [1] has shown that Ptolemy's inequality holds if a, b, c, d are any four points in a real inner-product space.

Definition 1.1 ([2])**.** *Let X be real normed linear space with norm $\|\cdot\|$. X is called Ptolemaic if for every $a, b, c, d \in X$ we have*

$$\|a-c\| \cdot \|b-d\| \leq \|a-b\| \cdot \|c-d\| + \|b-c\| \cdot \|a-d\| . \tag{1.2}$$

In this note, we establish some new interesting inequalities in Ptolemaic spaces.

Proposition 1.2. *Let a, b, c, d, e, f be elements of P, which is a Ptolemaic space. Then*

$$\begin{aligned} &\|a-b\| \cdot \|c-f\| \cdot \|d-e\| \\ &\qquad + \|a-b\| \cdot \|c-d\| \cdot \|e-f\| + \|b-c\| \cdot \|a-d\| \cdot \|e-f\| \\ &\qquad + \|b-c\| \cdot \|d-e\| \cdot \|f-a\| + \|c-d\| \cdot \|b-e\| \cdot \|f-a\| \\ &\qquad\qquad \geq \|a-d\| \cdot \|b-e\| \cdot \|c-f\| . \end{aligned} \tag{1.3}$$

Let's interpret the elements a, b, c, d, e, f as the vertices, in that order, of a hexagon. A hexagon has three diagonals, not two, and in fact, the product on the right of (1.3) is the product of the three diagonals. There are five products on the left. Two of these, $\|b-c\| \cdot \|d-e\| \cdot \|f-a\|$ and $\|a-b\| \cdot \|c-d\| \cdot \|e-f\|$ are the products of three sides that don't touch each other, and the other three are the products of a diagonal with the two sides that do not intersect.

Proof. We apply (1.2) for $\{a,b,c,d\}$, $\{b,c,d,e\}$ and we obtain that

$$\begin{aligned}
&\|a-b\|\cdot\|c-f\|\cdot\|d-e\| \\
&\qquad+(\|a-b\|\cdot\|c-d\|+\|b-c\|\cdot\|a-d\|)\cdot\|e-f\| \\
&\qquad+(\|b-c\|\cdot\|d-e\|+\|c-d\|\cdot\|b-e\|)\cdot\|f-a\| \\
&\qquad\quad\geq\|a-b\|\cdot\|c-f\|\cdot\|d-e\| \\
&\qquad+\|a-c\|\cdot\|b-d\|\cdot\|e-f\|+\|b-d\|\cdot\|c-e\|\cdot\|f-a\|\,.
\end{aligned}\tag{1.4}$$

Using (1.2) for $\{a,c,e,f\}$, $\{a,b,d,e\}$ on the right-hand side of (1.4) becomes

$$\begin{aligned}
&\|a-b\|\cdot\|c-f\|\cdot\|d-e\| \\
&\qquad\quad+\|a-c\|\cdot\|b-d\|\cdot\|e-f\|+\|b-d\|\cdot\|c-e\|\cdot\|f-a\| \\
&\qquad=\|a-b\|\cdot\|c-f\|\cdot\|d-e\| \\
&\qquad\quad+(\|a-c\|\cdot\|e-f\|+\|c-e\|\cdot\|f-a\|)\cdot\|b-d\| \\
&\qquad\geq\|a-b\|\cdot\|c-f\|\cdot\|d-e\|+\|a-e\|\cdot\|c-f\|\cdot\|b-d\| \\
&\qquad=(\|a-b\|\cdot\|d-e\|+\|a-e\|\cdot\|b-d\|)\cdot\|c-f\| \\
&\qquad\geq\|a-d\|\cdot\|b-e\|\cdot\|c-f\|\,.
\end{aligned}\tag{1.5}$$

From (1.4) and (1.5) we obtain (1.3). □

Example 1.3. *The equality holds for a regular hexagon. For example, if we take the six elements to be in the Euclidean plane and to be* $(\cos\frac{2\pi k}{6},\sin\frac{2\pi k}{6})$ *for* $0\leq k\leq 5$*. Each diagonal has length* 2 *and each side has length* 1*. The product of diagonals on the right is* 8*, the two products of* 3 *sides on the left are each* 1*, and the three products of* 2 *sides and a diagonal are each* 2*, so the left-hand side is also* 8*.*

If we use the ℓ^1 *norm in the plane, that is,* $\|(x,y)\|=|x|+|y|$*, and square the hexahedron by using the points*

$$a,b,c,d,e,f=(1,0),(1,1),(-1,1),(-1,0),(-1,-1),(1,-1)$$

we get that the horizontal diagonal is 2 *and the other two are* 4*, so the right-hand side is* 32*. The horizontal sides are* 2 *and the vertical sides are* 1*, so the two products of* 3 *sides are* 2*, the product with the horizontal diagonal is* 8*, and the other two products with diagonals are* 4 *each, so the left-hand side is* $2+2+8+4+4=20$*, which is less than the right-hand side of* 32*.*

Proposition 1.4. *Let* a,b,c *be distinct elements and* d *be an element of* P*, which is a Ptolemaic space. Then*

$$\frac{\|d-b\|\cdot\|d-c\|}{\|a-b\|\cdot\|a-c\|}+\frac{\|d-c\|\cdot\|d-a\|}{\|b-c\|\cdot\|b-a\|}+\frac{\|d-a\|\cdot\|d-b\|}{\|c-a\|\cdot\|c-b\|}\geq 1.\tag{1.6}$$

Proof. We apply (1.2) for $\{d,b,d+b-a,c\}$ and we obtain that

$$\|b-c\|\cdot\|b-a\|\leq\|d-b\|\cdot\|d+b-a-c\|+\|d-c\|\cdot\|d-a\|\,.\tag{1.7}$$

Using (1.2) for $\{c, b, d+b-a, b+c-a\}$ we obtain

$$\|d+b-a-c\| \cdot \|c-a\| \leq \|b-c\| \cdot \|d-c\| + \|d-a\| \cdot \|b-a\| . \tag{1.8}$$

Multiplying (1.7) by $\|c-a\|$ and substituting (1.8) we obtain

$$\begin{aligned} \|b-c\| \cdot \|b-a\| \cdot \|c-a\| &\leq \|c-a\| \cdot \|d-c\| \cdot \|d-a\| \\ &\quad + \|d-b\| \cdot \|b-c\| \cdot \|d-c\| + \|d-b\| \cdot \|d-a\| \cdot \|b-a\| , \end{aligned} \tag{1.9}$$

then dividing through by the left-hand side gives us

$$1 \leq \frac{\|d-b\| \cdot \|d-c\|}{\|a-b\| \cdot \|a-c\|} + \frac{\|d-c\| \cdot \|d-a\|}{\|b-c\| \cdot \|b-a\|} + \frac{\|d-a\| \cdot \|d-b\|}{\|c-a\| \cdot \|c-b\|}.$$

□

We can interpret the four elements as the vertices of a tetrahedron. If we use the inequality (1.9) we have four products, each of which is the product of three terms, and in each case the three terms are the edges of one face of the tetrahedron. So our conclusion is that each product is less than or equal to the sum of the other three.

Example 1.5. *The equality holds for a "flattened" tetrahedron in the plane, with the elements*

$$a, b, c, d = (1,0), \left(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right), \left(\cos\frac{4\pi}{3}, \sin\frac{4\pi}{3}\right), (0,0).$$

The face abc has sides of $\sqrt{3}$*, so its product is* $3\sqrt{3}$*. The other three faces have sides* $1, 1,$ *and* $\sqrt{3}$*, so each face product is* $\sqrt{3}$ *and the sum of all three is* $3\sqrt{3}$*. With the* ℓ^1 *norm we can use the elements*

$$a, b, c, d = (1,0), (0,1), (0,-1), (0,0)$$

so that the face abc has sides $2, 2, 2$ *and product* 8*, while each small face has sides* $2, 1, 1$ *and product* 2 *for a total of* 6*.*

Proposition 1.6. *Let* a, b, c *be distinct elements and* d *be an element of* P*, which is a Ptolemaic space and let* x, y, z *be real numbers such that* $x+y+z \geq 0, xy+yz+zx \geq 0$*. Then*

$$(y+z) \cdot \frac{\|d-a\|}{\|b-c\|} + (z+x) \cdot \frac{\|d-b\|}{\|c-a\|} + (x+y) \cdot \frac{\|d-c\|}{\|a-b\|} \geq 2\sqrt{xy+yz+zx}. \tag{1.10}$$

If $x = y = z = 1$, then inequality (1.10) is equivalent to

$$\begin{aligned} &\|a-d\| \cdot \|a-c\| \cdot \|a-b\| + \|b-d\| \cdot \|b-c\| \cdot \|a-b\| \\ &\quad + \|c-d\| \cdot \|a-c\| \cdot \|b-c\| \geq \sqrt{3}\|a-b\| \cdot \|b-c\| \cdot \|a-c\|. \end{aligned} \tag{1.11}$$

So we can write (1.11) as

$$ad \cdot ac \cdot ab + bd \cdot bc \cdot ab + cd \cdot ac \cdot bc \geq \sqrt{3} \cdot ab \cdot bc \cdot ac \tag{1.12}$$

for a tetrahedron with vertices $\{a, b, c, d\}$, where ad, ac, etc., denotes distances, respectively. The geometric interpretation is clear from (1.12).

To prove this result we need the following lemma.

Lemma 1.7. *Let $\alpha, \beta, \gamma, x, y, z$ be real numbers such that $\alpha + \beta + \gamma \geq 0, \alpha\beta + \beta\gamma + \gamma\alpha \geq 0, x + y + z \geq 0$, and $xy + yz + zx \geq 0$. Then*

$$(\beta+\gamma)x+(\gamma+\alpha)y+(\alpha+\beta)z \geq 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(xy + yz + zx)}. \tag{1.13}$$

Proof. Using the Cauchy-Schwarz inequality twice, we have

$$\begin{aligned}
&(\beta + \gamma)x + (\gamma + \alpha)y + (\alpha + \beta)z \\
&= (\alpha + \beta + \gamma)(x + y + z) - (\alpha x + \beta y + \gamma z) \\
&= \sqrt{(\alpha + \beta + \gamma)^2}\sqrt{(x + y + z)^2} - (\alpha x + \beta y + \gamma z) \\
&= \sqrt{2(\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha^2 + \beta^2 + \gamma^2)}\sqrt{2(xy + yz + zx) + (x^2 + y^2 + z^2)} \\
&- (\alpha x + \beta y + \gamma z) \\
&\geq \sqrt{2(\alpha\beta + \beta\gamma + \gamma\alpha)}\sqrt{2(xy + yz + zx)} + \sqrt{\alpha^2 + \beta^2 + \gamma^2}\sqrt{x^2 + y^2 + z^2} \\
&- (\alpha x + \beta y + \gamma z) \\
&= 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(xy + yz + zx)} + \sqrt{(\alpha^2 + \beta^2 + \gamma^2)(x^2 + y^2 + z^2)} \\
&- (\alpha x + \beta y + \gamma z) \\
&\geq 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(xy + yz + zx)}.
\end{aligned}$$

The lemma is proved. □

Proof of Proposition 1.6. Putting $\alpha = \frac{\|d-a\|}{\|b-c\|}, \beta = \frac{\|d-b\|}{\|c-a\|}, \gamma = \frac{\|d-c\|}{\|a-b\|}$ in the Lemma 1.7, we get

$$\begin{aligned}
&(y + z) \cdot \frac{\|d - a\|}{\|b - c\|} + (z + x) \cdot \frac{\|d - b\|}{\|c - a\|} + (x + y) \cdot \frac{\|d - c\|}{\|a - b\|} \\
&\geq 2\sqrt{(xy + yz + zx)} \\
&\times \sqrt{\left(\frac{\|d - a\|}{\|b - c\|} \cdot \frac{\|d - b\|}{\|c - a\|} + \frac{\|d - b\|}{\|c - a\|} \cdot \frac{\|d - c\|}{\|a - b\|} + \frac{\|d - c\|}{\|a - b\|} \cdot \frac{\|d - a\|}{\|b - c\|}\right)}.
\end{aligned} \tag{1.14}$$

Combining inequalities (1.6) and (1.14) leads to inequality (1.10) immediately. □

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References

[1] I. J. Schoenberg, *On metric arcs of vanishing Menger curvature*, Ann. of Math., **41** (1940), 715–726.

[2] M. S. Klamkin and A. Meir, *Ptolemy's inequality, chordal metric, multiplicative metric*, Pacific J. Math., **101.2** (1982), 389–392.

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