PERTURBATION ANALYSIS FOR THE DRAZIN INVERSE UNDER STABLE PERTURBATION IN BANACH SPACE

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Abstract. Let X be Banach space and let $T, \bar{T} = T + \delta T$ be bounded linear operators on X. Suppose that T has the Drazin inverse T^D and $\operatorname{Ind}(T) = n$. In this paper, we show that if $\|\delta T\|$ is sufficiently small and $\operatorname{Ran}(\bar{T}^n) \cap \operatorname{Ker}((T^D)^n) = \{0\}$, then $\bar{T}$ is Drazin invertible with $\operatorname{Ind}(\bar{T}) \leq n$. In this case, the expression of $\bar{T}^D$ is given and the upper bounds of $\|\bar{T}^D\|$ and

$$\frac{\|\bar{T}^D - T^D\|}{\|T^D\|}$$

are established. If $\dim X < \infty$, replacing $\operatorname{Ran}(\bar{T}^n) \cap \operatorname{Ker}((T^D)^n) = \{0\}$ by $\operatorname{rank}(\bar{T}^n) = \operatorname{rank}(T^n)$, we obtain the same perturbation results of the Drazin invertible matrix T as in the case of $\dim X = \infty$.

1. Introduction. Throughout the paper, $(X, \|\cdot\|)$ is a Banach space over the field $\mathbb{C}$ and $B(X)$ is the set of all bounded linear operators T on X. For $T \in B(X)$, we write $\operatorname{Ran}(T)$ (resp. $\operatorname{Ker}(T)$) to denote the range (resp. null space) of T. A nonzero operator T in $B(X)$ is said to be generalized invertible if there is $A \in B(X)$ such that $TAT = T,\ ATA = A$. Then A is called the generalized inverse of T, denoted by T^+. If X is a Hilbert space, T^+ is required to satisfy

$$TT^+T = T,\ T^+TT^+ = T^+,\ (T^+T)^* = T^+T,\ (TT^+)^* = TT^+. \tag{1.1}$$

In this situation, T^+ is called the Moore-Penrose inverse of T [11]. Recall from [6] that $T \in B(X)\backslash\{0\}$ is Drazin invertible, if there is an $A \in B(X)$ and a natural number k such that

$$T^kAT = T^k,\ ATA = A,\ AT = TA. \tag{1.2}$$

The least k such that (1.2) holds for some A is called the index of T, denoted by $\operatorname{Ind}(T) = k$. In this case, the A in (1.2) is called the Drazin inverse of T. We denote it by T^D. When $\operatorname{Ind}(T) = 1$, T^D is called the group inverse of T. We use the symbol $T^\#$ to denote it. Put $\operatorname{RG}(X) = \{T \in B(X) \mid T^+ \text{ exists}\}$ and

$$\operatorname{DI}(X) = \{T \in B(X) \mid T^D \text{ exists}\}, \quad \operatorname{GI}(X) = \{T \in \operatorname{DI}(X) \mid \operatorname{Ind}(T) = 1\}.$$

Let $T \in B(X)$ and $\bar{T} = T + \delta T$ be the small perturbation of T by $\delta T \in B(X)$. The perturbation theory of generalized inverse (or Drazin inverse) is concerned with the question that if $T \in \mathrm{RG}(X)$ (or $\mathrm{DI}(X)$), then when is $\bar{T}$ in $\mathrm{RG}(X)$ (or $\mathrm{DI}(X)$)? What are the upper bounds of $\|\bar{T}^{\gamma}\|$ and

$$\frac{\|\bar{T}^{\gamma} - T^{\gamma}\|}{\|T^{\gamma}\|},$$

where the symbol γ is $+$ or D? When X is a finite dimensional Hilbert space, $\|T^+\|\|\delta T\| < 1$ and Rank $(\bar{T})$ = Rank (T) (rank-preserving perturbation), we have

$$\|\bar{T}^+\| \leq \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|} \; \frac{\|\bar{T}^+ - T^+\|}{\|T^+\|} \leq \frac{1+\sqrt{5}}{2} \; \frac{\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|} \tag{1.3}$$

[13]. If X is a finite dimensional Banach space and $T, \bar{T} \in B(X)$ with Rank $(\bar{T}^l)$ = Rank (T^l), then the upper bounds of $\|T^D\|$ and

$$\frac{\|\bar{T}^D - T^D\|}{\|T^D\|}$$

have also been obtained recently in [14], where $l = \max\{\mathrm{Ind}(\bar{T}), \mathrm{Ind}(T)\}$.

When $X = \infty$, we need a new notation which can replace the rank-preserving perturbation in matrix theory. Let $T \in \mathrm{RG}(X)$ and $\bar{T} = T + \delta T \in B(X)$. Recall that $\bar{T}$ is the stable perturbation of T, if $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker}(T^+) = \{0\}$ (or equivalently, $\mathrm{Ran}(\bar{T}) \cap [\mathrm{Ran}(T)]^{\perp} = \{0\}$, when X is a Hilbert space) [3] and [15]. It is proved in [3] that $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker}(T^+) = \{0\}$ if and only if $\mathrm{Rank}(\bar{T}) = \mathrm{Rank}(T)$ when $\dim X < \infty$ and $\|T^+\|\|\delta T\| < 1$. Moreover, some conditions to characterize the stable perturbation of operators in Hilbert spaces and Banach spaces have been obtained by J. Ding in [5]. Using this notion, the authors in [2], [4], and [15] showed that (1.3) also holds when X is a Hilbert space, $\|T^+\|\|\delta T\| < 1$ and $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker}(T^+) = \{0\}$; when X is a general Banach space, $T \in \mathrm{RG}(X)$ and $\bar{T} = T + \delta T$ with

$$\|T^+\|\|\delta T\| < \frac{1}{1 + \|I - TT^+\|},$$

then $\bar{T} \in \mathrm{RG}(X)$ if and only if $\bar{T}$ is the stable perturbation of T if and only if $(I + \delta T T^+)^{-1}\bar{T}$ maps $\mathrm{Ker}(T)$ into $\mathrm{Ran}(T)$ [3]. This result generalized a famous theorem of Nashed's in [11].

As to the perturbation analysis of the Drazin invertible operators on X, there are also some results concerning the estimation of $\|\bar{T}^D\|$ and

$$\frac{\|\bar{T}^D - T^D\|}{\|T^D\|}$$

[7, 8, 10]. But we also notice that these results are based on the hypotheses that $\bar{T}$ is Drazin invertible and $\|\bar{T}\bar{T}^D - TT^D\|$ is small enough. Thus, the problem is: how can we guarantee $\bar{T} \in \mathrm{DI}(X)$ and $\|\bar{T}\bar{T}^D - TT^D\|$ is sufficiently small? These two problems have been solved in this paper in terms of the stable perturbation of bounded linear operators. Our main result of the paper is the following.

Let $T \in \mathrm{DI}(X)$ with $\mathrm{Ind}(T) = n$ and $\bar{T} = T + \delta T \in B(X)$ with

$$\kappa_D^n(T)\epsilon_T < \frac{1}{(2^n - 1)(1 + \|T^\pi\|)}.$$

If $\mathrm{Ran}(\bar{T}^n) \cap \mathrm{Ker}(T^D)^n = \{0\}$, then $\bar{T} \in \mathrm{DI}(X)$ with $\mathrm{Ind}(\bar{T}) \le n$, where

$$\kappa_D(T) = \|T\|\|T^D\|, \quad \epsilon_T = \frac{\|\delta T\|}{\|T\|}, \quad \text{and} \quad T^\pi = I - TT^D.$$

In this case, the upper bounds of $\|\bar{T}^D\|$ and

$$\frac{\|\bar{T}^D - T^D\|}{\|T^D\|}$$

are given.

2. Some Lemmas. Let $T \in \mathrm{DI}(X)$ with $\mathrm{Ind}(T) = k$ and put $T^\pi = I - TT^D$. Then $T^\pi T = TT^\pi$ and $T^D T^\pi = T^\pi T^D = 0$. Thus,

$$T_D = T\big|_{(I - T^\pi)X}$$

is invertible in $B((I-T^\pi)X)$ with the inverse

$$T_D^{-1} = T^D\big|_{(I-T^\pi)X} \quad \text{and} \quad T_N = T\big|_{T^\pi X}$$

is a nilpotent operator in $B(T^\pi X)$ with $T_N^k = 0$. Therefore, $T^D = (T^l)^\# T^{l-1}$ and $(T^D)^l = (T^l)^\#$, for all $l \geq k$. Conversely, we have the following.

Lemma 2.1. Let $T \in B(X)$ with $T^n \in \mathrm{GI}(X)$ for some $n \geq 1$. Then $T \in \mathrm{DI}(X)$ with $\mathrm{Ind}(T) \leq n$.

Proof. If T^n is group invertible, then by [9], T is invertible or 0 is an isolated point of T. Hence, $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi-nilpotent. Then $T^n = T_1^n \oplus T_2^n$, and the group inverse of T^n is given by $T_1^{-n} \oplus 0$. Hence, $A = T_1^{-1} \oplus 0$ satisfies the definition of the Drazin inverse with the index $\leq n$.

Lemma 2.2. Let $T \in \mathrm{RG}(X)$ and $\bar{T} = T + \delta T \in B(X)$ with

$$\|T^+\|\|\delta T\| < \frac{1}{1 + \|I - TT^+\|}.$$

Then $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker}(T^+) = \{0\}$ if and only if

$$(I - TT^+)\delta T(I - T^+T) = (I - TT^+)\delta T(I + T^+\delta T)^{-1}T^+\delta T(I - T^+T). \quad (2.1)$$

Proof. By [3], $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker} T^+ = \{0\}$ if and only if

$$(I - TT^+)\delta T(I + \delta T T^+)^{-1}\bar{T}(I - T^+T) = 0. \quad (2.2)$$

Since $(I + T^+\delta T)^{-1}T^+ = T^+(I + \delta T T^+)^{-1}$, it follows that

$$\begin{aligned}
&(I - TT^+)\delta T(I + T^+\delta T)^{-1}T^+\delta T(I - T^+T) \\
&\quad = (I - TT^+)[I + \delta T T^+ - I](I + \delta T T^+)^{-1}\delta T(I - T^+T) \\
&\quad = (I - TT^+)\delta T(I - T^+T) - (I - TT^+)\delta T(I + \delta T T^+)^{-1}\bar{T}(I - T^+T)
\end{aligned}$$

so (2.1) is equivalent to (2.2).

Lemma 2.3. Let $T \in \mathrm{GI}(X)$ and $\bar{T} = T + \delta T \in B(X)$ with

$$\kappa_{\#}(T)\epsilon_T < \frac{1}{1 + \|T^\pi\|}, \quad \text{where} \quad \kappa_{\#}(T) = \|T\|\|T^{\#}\|.$$

Then

$$\Phi(T) = I + \delta T(I - TT^{\#})\delta T[(I + T^{\#}\delta T)^{-1}T^{\#}]^2 \tag{2.3}$$

is invertible in $B(X)$ and

$$\|\Phi^{-1}(T)\| \leq \frac{(1 - \kappa_{\#}(T)\epsilon_T)^2}{(1 - \kappa_{\#}(T)\epsilon_T)^2 - \|T^\pi\|(\kappa_{\#}(T)\epsilon_T)^2}.$$

Proof. We have

$$\|(I + T^{\#}\delta T)^{-1}\| \leq \frac{1}{1 - \kappa_{\#}(T)\epsilon_T}$$

and

$$\|I - \Phi(T)\| \leq \|T^\pi\| \frac{(\kappa_{\#}(T)\epsilon_T)^2}{(1 - \kappa_{\#}(T)\epsilon_T)^2} < \frac{1}{\|T^\pi\|} \leq 1.$$

Thus, $\Phi(T)$ is invertible in $B(X)$ and

$$\|\Phi^{-1}(T)\| \leq \frac{1}{1 - \|I - \Phi(T)\|} \leq \frac{(1 - \kappa_{\#}(T)\epsilon_T)^2}{(1 - \kappa_{\#}(T)\epsilon_T)^2 - \|T^\pi\|(\kappa_{\#}(T)\epsilon_T)^2}.$$

Let $T \in \mathrm{GI}(X)$ and $\bar{T} = T + \delta T \in B(X)$. Then with respect to the decomposition, $X = (I - T^\pi)X + T^\pi X$, we have

$$T = \begin{bmatrix} T_{\#} & 0 \\ 0 & 0 \end{bmatrix}, \quad T^{\#} = \begin{bmatrix} T_{\#}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \delta T = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix},$$

where, $\delta_1 = (I - T^\pi)\delta T(I - T^\pi)$, $\delta_2 = (I - T^\pi)\delta T T^\pi$, $\delta_3 = T^\pi \delta T(I - T^\pi)$, and $\delta_4 = T^\pi \delta T T^\pi$. Let I_1 (resp. I_2) denote the identity operator on $(I - T^\pi)X$ (resp. $T^\pi X$). Then

$$\bar{T} = \begin{bmatrix} T_\# + \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix}, \quad I + T^\# \delta T = \begin{bmatrix} I_1 + T_\# \delta_1 & \delta_2 \\ 0 & I_2 \end{bmatrix}.$$

So if

$$\kappa_\#(T)\epsilon_T < \frac{1}{1 + \|T^\pi\|},$$

then $I_1 + T_\# \delta_1$ is invertible and

$$(I + T^\# \delta T)^{-1} T^\# = \begin{bmatrix} (I_1 + T_\# \delta_1)^{-1} T_\#^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and moreover, $I_1 + \delta_2 \delta_3 [(I_1 + T_\#^{-1} \delta_1)^{-1} T^\#]^2$ is also invertible, since

$$\Phi(T) = \begin{bmatrix} I_1 + \delta_2 \delta_3 [(I_1 + T_\#^{-1} \delta_1)^{-1} T^\#]^2 & 0 \\ \delta_4 \delta_3 [(I_1 + T_\#^{-1} \delta_1)^{-1} T^\#]^2 & I_2 \end{bmatrix}$$

is invertible by Lemma 2.3.

<u>Lemma 2.4</u>. Let $T \in \mathrm{GI}(X)$ and $\bar{T} = T + \delta T \in B(X)$ with

$$\kappa_\#(T)\epsilon_T < \frac{1}{1 + \|T^\pi\|}.$$

Put $C(T) = T^\pi \delta T (I + T^\# \delta T)^{-1} T^\#$, $D(T) = (I + T^\# \delta T)^{-1} T^\# \Phi^{-1}(T)$. Then $\bar{T} \in \mathrm{GI}(X)$ with

$$\bar{T}^\# = (I + C(T))(D(T) + D^2(T)\delta T T^\pi)(I - C(T)) \qquad (2.4)$$

if and only if $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker}(T^\#) = \{0\}$.

Proof. Let $\delta_1, \dots, \delta_4$ and $T_\#$ be as above. Put $\Delta = \delta_4 - \delta_3(I_1 + T_\#^{-1}\delta_1)^{-1}T_\#^{-1}\delta_2$. It is easy to check that $(I + C(T)))(I - C(T)) = (I - (T))(I + (T)) = I$,

$$I \pm C(T) = \begin{bmatrix} I_1 & 0 \\ \pm\delta_3(I_1 + T_\#^{-1}\delta_1)^{-1}T_\#^{-1} & I_2 \end{bmatrix},$$

and (2.1) is equivalent to $\Delta = 0$ when replacing T^+ by $T^\#$ in (2.1). Notice that

$$(I - C(T))\bar{T}(I + C(T)) = \begin{bmatrix} T_\# + \delta_1 + \delta_2\delta_3(T_\# + \delta_1)^{-1} & \delta_2 \\ \Delta\delta_3(T_\# + \delta_1)^{-1} & \Delta \end{bmatrix} = \bar{T}_0. \quad (2.5)$$

So if $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker}(T^\#) = \{0\}$, then

$$\bar{T}_0 = \begin{bmatrix} T_\# + \delta_1 + \delta_2\delta_3(T_\# + \delta_1)^{-1} & \delta_2 \\ 0 & 0 \end{bmatrix}$$

and

$$\bar{T}_0^\# = \begin{bmatrix} T_\# + \delta_1 + \delta_2\delta_3(T_\# + \delta_1)^{-1}]^{-1} & T_\# + \delta_1 + \delta_2\delta_3(T_\# + \delta_1)^{-1}]^{-2}\delta_2 \\ 0 & 0 \end{bmatrix}$$

$$= D(T) + D^2(T)\delta T T^\pi.$$

Consequently, $\bar{T}^\# = (I + C(T))(D(T) + D^2(T)\delta T T^\pi)(I - C(T))$.

On the other hand, if $\bar{T} \in \mathrm{GI}(X)$ and $\bar{T}^\#$ has the expression (2.4), then by (2.5), $\bar{T}_0^\# = D(T) + D^2(T)\delta T T^\pi$. From $\bar{T}_0\bar{T}_0^\#\bar{T}_0 = \bar{T}_0$, we get that $\Delta = 0$. Thus, $\mathrm{Ran}(\bar{T}) \cap \mathrm{Ker}(T^\#) = \{0\}$ by Lemma 2.2.

3. Perturbation Analysis for Drazin Inverse. We first consider the perturbation of group inverse under stable perturbation. We have the following theorem.

Theorem 3.1. Let $T \in \mathrm{GI}(X)$ and $\bar{T} = T + \delta T \in B(X)$ with

$$\kappa_\#(T)\epsilon_T < \frac{1}{1 + \|T^\pi\|}.$$

Assume that $\bar{T}$ is the stable perturbation of T. Then $\bar{T} \in \mathrm{DI}(X)$ and

$$\|\bar{T}^{\#}\| \leq \frac{\|T^{\#}\|}{[1-(1+\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T]^2},$$

$$\|\bar{T}^{\pi} - T^{\pi}\| < \frac{2\|T^{\pi}\|\kappa_{\#}(T)\epsilon_T}{1-(1+\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T},$$

$$\frac{\|\bar{T}^{\#} - T^{\#}\|}{\|T^{\#}\|} \leq \frac{(1+2\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T}{[1-(1+\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T]^2}.$$

Proof. We keep $C(T)$, $D(T)$ in Lemma 2.4. Then by Lemma 2.4, $\bar{T} \in \mathrm{DI}(X)$. Since $D(T)(I - C(T)) = D(T)$, it follows from (2.4) that

$$\bar{T}^{\#} = (I + C(T))D(T) + (I + C(T))D^2(T)\delta T T^{\pi}(I - C(T)) \quad (3.1)$$

$$\|\bar{T}^{\#}\| \leq (1 + \|C(T)\|)\|D(T)\| + (1 + \|C(T)\|)^2\|D(T)\|^2\|\delta T\|\|T^{\pi}\|. \quad (3.2)$$

We now estimate $1 + \|C(T)\|$ and $\|D(T)\|$, respectively. We have

$$1 + \|C(T)\| \leq 1 + \|T^{\pi}\|\frac{\kappa_{\#}(T)\epsilon_T}{1-\kappa_{\#}(T)\epsilon_T} = \frac{1 + (\|T^{\pi}\| - 1)\kappa_{\#}(T)\epsilon_T}{1-\kappa_{\#}(T)\epsilon_T}$$

$$\|D(T)\| \leq \frac{(1-\kappa_{\#}(T)\epsilon_T)\|T^{\#}\|}{[1-(1+\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T][1 + (\|T^{\pi}\| - 1)\kappa_{\#}(T)\epsilon_T]}.$$

Thus,

$$(1 + \|C(T)\|)^2\|D(T)\|^2\|\delta T\|\|T^{\pi}\| \leq \frac{\|T^{\#}\|\|T^{\pi}\|\kappa_{\#}(T)\epsilon_T}{[1-(1+\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T]^2}$$

and hence, by (3.2),

$$\|\bar{T}^{\#}\| \leq \frac{(1-\kappa_{\#}(T)\epsilon_T)\|T^{\#}\|}{[1-(1+\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T]^2} \leq \frac{\|T^{\#}\|}{[1-(1+\|T^{\pi}\|)\kappa_{\#}(T)\epsilon_T]^2}.$$

By (2.4) and (2.5), $\bar{T}\bar{T}^{\#} = (I + C(T))[I - T^\pi + D(T)\delta T T^\pi](I - C(T))$. So

$$\begin{aligned}\|\bar{T}^\pi - T^\pi\| &\le \|C(T)(I - T^\pi)\| + \|(I + C(T))D(T)\delta T T^\pi (I - C(T))\| \\ &\le \|C(T)\| + (1 + \|C(T)\|)^2 \|D(T)\| \|\delta T\| \|T^\pi\| \\ &\le \frac{2\|T^\pi\| \kappa_\#(T)\epsilon_T}{1 - (1 + \|T^\pi\|)\kappa_\#(T)\epsilon_T}.\end{aligned}$$

Finally, by (3.1)

$$\begin{aligned}&\|\bar{T}^\# - T^\#\| \\ &\le \|D(T) - T^\#\| + \|C(T)D(T)\| + (1 + \|C(T)\|)^2 \|D(T)\|^2 \|\delta T\| \|T^\pi\|.\end{aligned}$$

Now, using Lemma 2.2 and the fact that

$$(\kappa_\#(T)\epsilon_T)^2 \le \kappa_\#(T)\epsilon_T < 1$$

we have

$$\begin{aligned}\|D(T) - T^\#\| &= \|[(I + T^\pi \delta T)^{-1} T^\# - T^\#]\Phi^{-1}(T) + T^\#(\Phi^{-1}(T) - I)\| \\ &\le \frac{T^\# \kappa_\#(T)\epsilon_T}{1 - \kappa_\#(T)\epsilon_T} \|\Phi^{-1}(T)\| + \|T^\#\| \|\Phi^{-1}(T)\| \|\Phi(T) - I\| \\ &\le \frac{\|T^\#\| \kappa_\#(T)\epsilon_T}{1 - (1 + \|T^\pi\|)\kappa_\#(T)\epsilon_T} \\ \|C(T)D(T)\| &\le \frac{\|T^\#\| \|T^\pi\| \kappa_\#(T)\epsilon_T}{1 - (1 + \|T^\pi\|)\kappa_\#(T)\epsilon_T}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\bar{T}^\# - T^\#\| \\ &\le \frac{\|T^\#\| \kappa_\#(T)\epsilon_T}{1 - (1 + \|T^\pi\|)\kappa_\#(T)\epsilon_T} \left[1 + \|T^\pi\| + \frac{\|T^\pi\|}{1 - (1 + \|T^\pi\|)\kappa_\#(T)\epsilon_T}\right] \\ &\le \frac{(1 + 2\|T^\pi\|)\|T^\#\| \kappa_\#(T)\epsilon_T}{[1 - (1 + \|T^\pi\|)\kappa_\#(T)\epsilon_T]^2}.\end{aligned}$$

Let T, $\bar{T} = T + \delta T \in B(X)$ and $\delta T^j = (T + \delta T)^j - T^j$, $j = 1, \dots, n$. Then

$$\|\delta T^j\| \leq (\|T\| + \|\delta T\|)^{j-1}\|\delta T\| + \|T\|\|\delta T^{j-1}\|. \tag{3.3}$$

Suppose $\epsilon_T < 1$. From (3.3), we can deduce that

$$\|\delta T^n\| \leq \|\delta T\| \sum_{j=0}^{n-1} \|T\|^j (\|T\| + \|\delta T\|)^{n-1-j}$$

$$= \|T\|^n \epsilon_T \sum_{j=0}^{n-1} (1 + \epsilon_T)^j < (2^n - 1)\|T\|^n \epsilon_T.$$

Now we present the main result of the paper as follows.

Theorem 3.2. Let $T \in \mathrm{DI}(X)$ with $\mathrm{Ind}(T) = n$ and $\bar{T} = T + \delta T \in B(X)$ with

$$\kappa_D^n(T)\epsilon_T < \frac{1}{(2^n - 1)(1 + \|T^\pi\|)}.$$

Assume that $\mathrm{Ran}(\bar{T}^n) \cap \mathrm{Ker}(T^D)^n = \{0\}$. Then $\bar{T} \in \mathrm{DI}(X)$ with $\mathrm{Ind}(\bar{T}) \leq n$ and

$$\bar{T}^D = (I + C(T^n))(D(T^n) + D^2(T^n)\delta T^n T^\pi)(I + C(T^n))(T + \delta T)^{n-1}, \tag{3.4}$$

$$\|\bar{T}^D\| \leq \frac{2^{n-1}\kappa_D^{n-1}(T)\|T^D\|}{[1 - (2^n - 1)(1 + \|T^\pi\|)\kappa_D^n(T)\epsilon_T]^2},$$

$$\|\bar{T}^\pi - T^\pi\| \leq \frac{2(2^n - 1)\kappa_D^n(T)\epsilon_T\|T^\pi\|}{1 - (2^n - 1)(1 + \|T^\pi\|)\kappa_D^n(T)\epsilon_T}$$

$$\frac{\|\bar{T}^D - T^D\|}{\|T^D\|} \leq \frac{2^{n-1}(2^n - 1)(1 + 2\|T^\pi\|)\kappa_D^{2n-1}(T)\epsilon_T}{[1 - (2^n - 1)(1 + \|T^\pi\|)\kappa_D^n(T)\epsilon_T]^2} + (2^{n-1} - 1)\kappa_D^n(T)\epsilon_T.$$

Proof. We have $T^n \in \mathrm{GI}(X)$ and $(T^D)^n = (T^n)^\#$, $T^n(T^n)^\# = I - T^\pi$. Noting that $\kappa_D(T) \geq \|TT^D\| \geq 1$, we have $\epsilon_T < 1$ and

$$\|\delta T^n\|\|(T^n)^\#\| < (2^n - 1)\kappa_D^n(T)\epsilon_T < \frac{1}{1 + \|T^\pi\|}. \tag{3.5}$$

Applying Theorem 3.1 to T^n and $\bar{T}^n = T^n + \delta T^n$, we get that $\bar{T}^n \in \mathrm{DI}(X)$ and

$$\|(\bar{T}^n)^\#\| \leq \frac{\|(T^n)^\#\|}{[1 - (1 + \|T^\pi\|)\|\delta T^n\|\|(T^n)^\#\|]^2} \tag{3.6}$$

$$\|\bar{T}^n(\bar{T}^n)^\# - T^n(T^n)^\#\| \leq \frac{2\|T^\pi\|\|\delta T^n\|\|(T^n)^\#\|}{1 - (1 + \|T^\pi\|)\|\delta T^n\|\|(T^n)^\#\|)} \tag{3.7}$$

$$\|(\bar{T}^n)^\# - (T^n)^\#\| \leq \frac{(1 + 2\|T^\pi\|)\|\delta T^n\|\|(T^n)^\#\|^2}{[1 - (1 + \|T^\pi\|)\|\delta T^n\|\|(T^n)^\#\|]^2}. \tag{3.8}$$

By Lemma 2.1, $\bar{T} \in \mathrm{DI}(X)$ with $\mathrm{Ind}(\bar{T}) \leq n$. So $\bar{T}^D = (\bar{T}^n)^\# \bar{T}^{n-1}$. Replacing T by T^n in (2.4), we obtain the expression of $(\bar{T}^n)^\#$ and hence, we get (3.4). Furthermore, $\|\bar{T}^D\| \leq \|(\bar{T}^n)^\#\|\|T\|^{n-1}(1 + \epsilon_T)^{n-1}$ and

$$\begin{aligned}
&\|\bar{T}^D - T^D\| = \|\|(\bar{T}^n)^\#\bar{T}^{n-1} - (T^n)^\# T^{n-1}\| \\
&\leq \|(\bar{T}^n)^\# - (\bar{T}^n)^\#\|\|T\|^{n-1}(1 + \epsilon_T)^{n-1} + \|(T^n)^\#\|\|\bar{T}^{n-1} - T^{n-1}\| \\
&< 2^{n-1}\|(\bar{T}^n)^\# - (\bar{T}^n)^\#\|\|T\|^{n-1} + (2^{n-1} - 1)\|T^D\|\kappa_D^{n-1}(T)\epsilon_T.
\end{aligned}$$

Combining these inequalities with (3.5), (3.6), (3.7), and (3.8), we can get the assertions easily.

Let V_1, V_2 be two closed subspaces of X. Put

$$\delta(V_1, V_2) = \sup\{\mathrm{dist}\ (x, V_2) |\, x \in V_1, \|x\| = 1\},$$

$$\hat{\delta}(V_1, V_2) = \max\{\delta(V_1, V_2), \delta(V_2, V_1)\}.$$

Suppose that $\{T_n\}_{n=0}^\infty \subset \mathrm{DI}(X)$ and

$$\lim_{n\to\infty} T_n = T_0.$$

Let $T_n = C_n + N_n$ be the core nilpotent decomposition of T_n, $n \geq 0$ [12]. Using $\hat{\delta}(\mathrm{Ran}(C_n), \mathrm{Ran}(C_0))$, Rakočević characterized

$$\lim_{n\to\infty} T_n^D = T_0^D.$$

Now we give a relatively simple condition such that

$$\lim_{n\to\infty} T_n^D = T_0^D$$

as follows.

Corollary 3.3. Let $T_n \in \mathrm{DI}(X)$, $(n \geq 0)$ with

$$\lim_{n\to\infty} T_n = T_0.$$

Suppose that

$$l = \sup_{n\geq 0} \mathrm{Ind}(T_n) < +\infty.$$

Then

$$\lim_{n\to\infty} T_n^D = T_0^D$$

if and only if $\mathrm{Ran}(T_n^l) \cap \mathrm{Ker}(T_0^l)^{\#} = \{0\}$ eventually.

Proof. We have $T_n^l = C_n^l \in \mathrm{GI}(X)$ and $\mathrm{Ran}(C_n^l) = \mathrm{Ran}(C_n)$, $n \geq 0$. If $\mathrm{Ran}(T_n^l) \cap \mathrm{Ker}(T_0^l)^{\#} = \{0\}$, then by Theorem 3.1,

$$\lim_{n\to\infty} \|(T_n^l)^{\#} - (T_0^l)^{\#}\| = 0.$$

Since $T_n^D = (T_n^l)^{\#}(T_n)^{l-1}$, it follows that

$$\lim_{n\to\infty} T_n^D = T_0^D.$$

Conversely, if

$$\lim_{n\to\infty} T_n^D = T_0^D,$$

then

$$\lim_{n\to\infty} \hat{\delta}(\mathrm{Ran}(C_n),\ \mathrm{Ran}(C_0)) = 0$$

by [12]. Since

$$\lim_{n\to\infty} \|T_n^l - T_0^l\| = 0,$$

it follows from [3] that $\mathrm{Ran}(T_n^l) \cap \mathrm{Ker}(T_0^l)^{\#} = \{0\}$ eventually.

We end the paper with the following remark.

Remark 3.4. Let $T \in \mathrm{DI}(X)$ with $\mathrm{Ind}(T) = n$ and $\bar{T} = T + \delta T \in B(X)$.

(1) When $\dim X < \infty$, Rank $(\bar{T}^n) = \mathrm{Rank}(T^n)$ and ϵ_T is sufficiently small, Theorem 3.2 gives perturbation results of Drazin invertible matrices.

(2) When $\dim X < \infty$, Corollary 3.3 is the equivalent condition of the continuity of Drazin invertible matrices given by Campbell and Meyer in [1].

(3) If we require $\bar{T} \in \mathrm{DI}(X)$ with $\mathrm{Ind}(\bar{T}) \leq n$ and

$$\lim_{\delta T \to 0} \bar{T}^D = T^D,$$

then by Corollary 3.3, $\mathrm{Ran}(\bar{T}^n) \cap \mathrm{Ker}(T^n)^{\#} = \{0\}$ when $\|\delta T\|$ is sufficiently small. So in this case, the expression of $\bar{T}^D$ given by (3.4) is unique. This means that Theorem 3.2 solves the problem of continuous perturbation of Drazin invertible operators or matrices.

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