No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

157. [2005, 194] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let a, x be real numbers such that $1 < a < x$. Prove that

$$\left(\sum_{k=1}^{n} \log^{-1}_{a^{k(k+1)}} x\right)\left(\log_a \sqrt[n+1]{x^n}\right) \geq n^2.$$

Solution by Joe Flowers, St. Mary's University, San Antonio, Texas. We have

$$\log_a \sqrt[n+1]{x^n} = \frac{n}{n+1} \log_a x$$

and

$$\log_{a^{k(k+1)}} x = \frac{\log_a x}{\log_a a^{k(k+1)}} = \frac{\log_a x}{k(k+1)},$$

so

$$\log^{-1}_{a^{k(k+1)}} x = \frac{k(k+1)}{\log_a x}.$$

Therefore,

$$\left(\sum_{k=1}^{n} \log^{-1}_{a^{k(k+1)}} x\right)\left(\log_a \sqrt[n+1]{x^n}\right) = \left(\frac{1}{\log_a x}\sum_{k=1}^{n} k(k+1)\right)\left(\frac{n}{n+1}\log_a x\right)$$

$$= \frac{n}{n+1}\left(\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right) = \frac{n}{n+1}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= n^2 \cdot \frac{n+2}{3} \geq n^2,$$

since

$$\frac{n+2}{3} \geq 1$$

with equality only when $n = 1$.

We note that the condition $1 < a < x$ may be replaced by the less restrictive condition that a and x can be any two positive reals provided neither is equal to one.

Also solved by Joe Howard, Portales, New Mexico; Ovidui Furdui (student), Western Michigan University, Kalamazoo, Michigan; Nina Shang and Huizeng Qin (jointly), Shandong University of Technology, Zibo, People's Republic of China; Kenneth B. Davenport, Dallas, Pennsylvania; Joe Dence, St. Louis, Missouri; and the proposer.

158. [2005; 194] *Proposed by Ovidui Furdui (student), Western Michigan University, Kalamazoo, Michigan.*

Find the sum:

$$\sum_{k=1}^{\infty} \left[(k+1)^2 \ln \frac{(k+1)^2}{k(k+2)} - 1 \right].$$

Solution by Russell Jay Hendel, Towson University, Towson, Maryland. The problem sum equals $1.5 - \ln(\pi) = 0.355\ldots.$

To prove this fix an integer $n \geq 2$ and define for integer $i \geq 2$,

$$s(i) = i^2 \ln\left(\frac{i^2}{i^2-1}\right) - 1 = 2i^2 \ln(i) - i^2 \ln(i-1) - i^2 \ln(i+1) - 1,$$

$$S(n) = \sum_{i=2}^{n+1} s(i) = -n + \sum c(i) \ln(i),$$

with $c(i)$ polynomial functions in i. Clearly the problem sum equals

$$\lim_{n \to \infty} S(n).$$

We claim

$$c(i) = \begin{cases} -1, & \text{if } i = 2, \\ -2, & \text{for } 3 \le i \le n, \\ (n+1)^2 + (2n+1), & \text{if } i = n+1, \\ -(n+1)^2, & \text{if } i = n+2, \\ 0, & \text{for } i > n+2. \end{cases} \tag{1}$$

The proof of (1) is straightforward. For example for $3 \le i \le n$ the contribution to $c(i)$ from $s(i)$, $s(i+1)$, and $s(i-1)$ respectively, is $2i^2$, $-(i+1)^2$, and $-(i-1)^2$ which sums to -2 as required. Proofs of the other cases of (1) are treated similarly.

It follows from (1) that

$$S(n) = \ln\left(\frac{1}{e^n}\frac{2}{n!^2}\left(\frac{n+1}{n+2}\right)^{(n+1)^2}(n+1)^{2n+1}\right). \tag{2}$$

To evaluate (2) as $n \to \infty$ we use the following formulae:

$$\begin{cases} n!^2 \sim (2\pi n)\left(\frac{n}{e}\right)^{2n}, & \text{Stirling's formula,} \\ \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n+1} = e^2 & \\ \left(\frac{n+1}{n+2}\right)^{(n+1)^2} \sim e^{-n-0.5} & \end{cases} \tag{3}$$

The last two equations follow by taking logarithms and using Taylor's formula. For example

$$\lim_{n\to\infty} \ln\left(e^n\left(\frac{n+1}{n+2}\right)^{(n+1)^2}\right) = \lim_{n\to\infty}\left(n - \frac{(n+1)^2}{n+2} - \frac{1}{2}\frac{(n+1)^2}{(n+2)^2} + o(1)\right) = -\frac{1}{2}$$

proving the last formula in (3).

Substituting the limits of (3) into (2) and performing some straightforward cancellations shows

$$S(n) \sim \ln\left(\frac{e^{1.5}}{\pi}\right).$$

This completes the proof.

Also solved by Joe Flowers, St. Mary's University, San Antonio, Texas; Joe Howard, Portales, New Mexico; Huizeng Qin and Nina Shang (jointly), Shandong University of Technology, Zibo, People's Republic of China; and the proposer. A partial solution was also received.

159. [2005; 195] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let m be a positive integer and let $A_1, A_2, \dots, A_m$ be $n \times n$ real symmetric matrices. Prove that

$$\det(A_1^2 + A_2^2 + \cdots + A_m^2) \geq 0.$$

Solution by Ovidiu Furdui (student), Western Michigan University, Kalamazoo, Michigan. Let $A = A_1^2 + A_2^2 + \cdots + A_m^2$, and let T be the operator defined on $\mathbb{R}^n$ whose matrix is A, i.e., $Tx = Ax$. We notice that T is a positive operator since for all $x \in \mathbb{R}^n$, $\langle Tx, x\rangle \geq 0$, where $\langle a, b\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ is the inner product on $\mathbb{R}^n$. A calculation shows that

$$\langle Tx, x\rangle = \langle Ax, x\rangle = \left\langle \sum_{i=1}^{m} A_i^2 x, x \right\rangle$$

$$= \sum_{i=1}^{m} \langle A_i^2 x, x\rangle = \sum_{i=1}^{m} \langle A_i x, A_i^* x\rangle,$$

where A_i^* is the adjoint of A_i. The matrix A_i is real and symmetric, hence $A_i^* = A_i$. It follows that

$$\langle Tx, x\rangle = \sum_{i=1}^{m} \langle A_i x, A_i x\rangle = \sum_{i=1}^{m} \|A_i x\|^2 \geq 0.$$

This implies that the operator T is positive. But since T is a positive operator, there exists a unique positive operator on $\mathbb{R}^n$, denoted by $T^{\frac{1}{2}}$ (which is also called the square root of T) such that $(T^{\frac{1}{2}})^2 = T$. If B is the matrix of $T^{\frac{1}{2}}$, we see that $T^{\frac{1}{2}}x = Bx$ and since $(T^{\frac{1}{2}})^2 = T$ we get that

$$B^2 = A = A_1^2 + A_2^2 + \cdots + A_m^2.$$

Therefore,

$$\det(A_1^2 + \cdots + A_m^2) = \det B^2 = (\det B)^2 \geq 0.$$

Also solved by Joe Flowers and Hernan Rivera (jointly), Texas Lutheran University, Seguin, Texas; Nina Shang and Huizeng Qin (jointly), Shandong University of Technology, Zibo, People's Republic of China; and the proposer.

160. [2005; 195] *Proposed by Zdravko F. Starc, 26300 Vršac, Serbia and Montenegro.*

Let F_n be the Fibonacci numbers defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that for $n \geq 1$,

$$\sqrt{F_1^4 + F_2^4 + F_3^4} + \sqrt{F_2^4 + F_3^4 + F_4^4} + \cdots + \sqrt{F_n^4 + F_{n+1}^4 + F_{n+2}^4}$$

$$= \frac{\sqrt{2}}{2}(2F_{n+1}^2 + F_{n+2}^2 + 3F_nF_{n+1} - 3).$$

Solution I by Joe Howard, Portales, New Mexico. Using Candido's Identity

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 = 2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)$$

and the identity

$$\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$$

the LHS becomes:

$$\frac{\sqrt{2}}{2}\big((F_1^2+F_2^2+F_3^2)+(F_2^2+F_3^2+F_4^2)+\cdots+(F_n^2+F_{n+1}^2+F_{n+2}^2)\big)$$

$$= \frac{\sqrt{2}}{2}\Bigg(\big(F_1^2+F_2^2+\cdots+F_n^2\big)+\big(F_2^2+F_3^2+\cdots+F_{n+1}^2\big)$$

$$+\big(F_3^2+F_4^2+\cdots+F_{n+2}^2\big)\Bigg)$$

$$= \frac{\sqrt{2}}{2}\Bigg(\big(F_nF_{n+1}\big)+\big(F_nF_{n+1}+F_{n+1}^2-F_1^2\big)$$

$$+\big(F_nF_{n+1}+F_{n+1}^2+F_{n+2}^2-F_1^2-F_2^2\big)\Bigg)$$

$$= \frac{\sqrt{2}}{2}\Bigg(3F_nF_{n+1}+2F_{n+1}^2+F_{n+2}^2-3\Bigg)$$

Solution II by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain. We will argue by mathematical induction. The case when $n=1$ trivially holds. Then, it suffices to see that

$$\sqrt{F_{n+1}^4+F_{n+2}^4+F_{n+3}^4}+\frac{\sqrt{2}}{2}\big(2F_{n+1}^2+F_{n+2}^2+3F_nF_{n+1}-3\big)$$

$$= \frac{\sqrt{2}}{2}\big(2F_{n+2}^2+F_{n+3}^2+3F_{n+1}F_{n+2}-3\big)$$

or equivalently,

$$\sqrt{F_{n+1}^4+F_{n+2}^4+F_{n+3}^4} = \frac{\sqrt{2}}{2}\big(2(F_{n+2}^2-F_{n+1}^2)+(F_{n+3}^2-F_{n+2}^2)+3F_{n+1}^2\big)$$

$$= \sqrt{2}\big(F_n^2+3F_{n+1}^2+3F_nF_{n+1}\big).$$

Squaring the LHS of the preceding identity, we have

$$\begin{aligned}
F_{n+1}^4 + F_{n+2}^4 + F_{n+3}^4 &= F_{n+1}^4 + (F_n + F_{n+1})^4 + (F_n + 2F_{n+1})^4 \\
&= 2\big(F_n^4 + 9F_{n+1}^4 + 15F_n^2F_{n+1}^2 + 6F_nF_{n+1}(F_n^2 + 3F_{n+1}^2)\big) \\
&= 2\big(F_n^2 + 3F_{n+1}^2 + 3F_nF_{n+1}\big)^2
\end{aligned}$$

and by the principle of mathematical induction the proof is complete.

Also solved by Tom Leong, Brooklyn, NY; Joe Flowers, St. Mary's University, San Antonio, Texas; Russell Jay Hendel, Towson University, Towson, Maryland; Ovidiu Furdui (student), Western Michigan University, Kalamazoo, Michigan; Jim Bruening, Southeast Missouri State University, Cape Girardeau, Missouri; Jerry Bergum, Brookings, South Dakota; Kenneth B. Davenport, Dallas, Pennsylvania; Huizeng Qin and Nina Shang (jointly), Shandong University of Technology, Zibo, People's Republic of China; and the proposer.