STRONG LAW OF LARGE NUMBERS FOR ARRAYS OF ROWWISE PAIRWISE NQD RANDOM VARIABLES

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Abstract. A strong law of large numbers for arrays of rowwise pairwise negatively quadrant dependent (NQD) random variables is obtained which relaxes the usual assumption of rowwise negatively dependent (ND). The moment conditions of the main result are weaker than the previous results. The result obtained generalizes and improves the result of Taylor (2002).

1. Introduction.

Definition 1. Two random variables X and Y are negatively quadrant dependent (NQD), if for all $x, y \in R$, we have

$$P(X < x, Y < y) \leq P(X < x)P(Y < y). \tag{1}$$

A random variables sequence $\{X_k, k \in N\}$ is said to be pairwise NQD, if for all $i \neq j$, X_i and X_j are NQD.

This concept was given by Lehmann [1]. In 2002, Taylor [2] gave the concept of LND, UND and ND. We have the following definitions.

Definition 2. A random variables sequence $\{X_k, k \in N\}$ is said to be

(a) lower negatively dependent (LND), if for each n

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i) \tag{2}$$

for all $x_1, \dots, x_n \in R$,

(b) upper negatively dependent (UND), if for each n

$$P(X_1 > x_1, \dots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i) \tag{3}$$

for all $x_1, \dots, x_n \in R$,

(c) negatively dependent (ND), if both (2) and (3) hold.

Taylor [2] gave a strong law of large numbers for arrays of rowwise ND random variables. It is obvious that pairwise NQD is more general than ND or NA. The concept of NA was given by Joag [4]. Shao [5] discussed moment inequality, complete convergence, invariance principle, and the strong law of large numbers. In this paper, a strong law of large numbers for arrays of rowwise pairwise NQD random variables is obtained which relaxes the assumption of rowwise ND in Taylor [2]. The moment conditions of the main result are weaker than those of Taylor [2]. The main result of this paper is the following theorem.

Theorem 1. Let $\{X_{nk},\ 1 \le k \le n,\ n \ge 1\}$ be an array of rowwise pairwise NQD random variables with $EX_{nk} = 0$ for each n and k. Let

$$S_n = \sum_{k=1}^{n} X_{nk}.$$

$$\sup_{n,k} P(|X_{nk}| > t) \le P(|X| > t)$$

for all $t \in R$. Also,

$$E|X|^p h(|X|^{\frac{1}{\alpha}}) < \infty, \tag{4}$$

where $\alpha p > 1$, $0 < p < 2$, $h(x) > 0$ is a slowly varying function when $x \to \infty$. When $\alpha \le 1$, $EX = 0$. Then,

$$\text{for all } \varepsilon > 0,\ \sum_{n=1}^{\infty} n^{p\alpha-2} h(n) P(\max_{1\le j\le n} |S_j| > \varepsilon n^{\alpha}) < \infty. \tag{5}$$

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n \ll b_n$ will mean $a_n \le C b_n$.

2. Proof of the Main Theorem. In order to prove our theorems, we need the following lemma.

Lemma 1 (Lehmann [1]). If X and Y are NQD, then

(i) $EXY \le EXEY$;

(ii) $P(X > x, Y > y) \le P(X > x)P(Y > y)$ for all $x, y \in R$;

(iii) if $f(x), g(x)$ are non-decreasing (or non-increasing) functions, then $f(X), g(X)$ are also NQD.

Lemma 2. Let $\{Y_i, i \geq 1\}$ be a sequence of centered pairwise NQD random variables and $E|Y_i|^2 < \infty$ for every $i \geq 1$. Then there exists a C, such that

$$E\left|\sum_{i=1}^{n} Y_i\right|^2 \leq \sum_{i=1}^{n} E|Y_i|^2$$

$$E \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} Y_i\right|^2 \leq C \log^2 n \sum_{i=1}^{n} E|Y_i|^2.$$

Proof of Lemma 2. By Lemma 1 and $EY_i = 0$, we have

$$E\left|\sum_{i=1}^{n} Y_i\right|^2 \leq \sum_{i=1}^{n} E|Y_i|^2 + 2 \sum_{1\leq i<j\leq n} EY_i EY_j \leq \sum_{i=1}^{n} E|Y_i|^2.$$

By

$$E\left|\sum_{i=1}^{n} Y_i\right|^2 \leq \sum_{i=1}^{n} E|Y_i|^2$$

and Theorem 2.4.1 in Stout [3], we have

$$E \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} Y_i\right|^2 \leq C\left(\frac{\log(2n)}{\log 2}\right)^2 \sum_{i=1}^{n} E|Y_i|^2 \leq C \log^2 n \sum_{i=1}^{n} E|Y_i|^2.$$

Proof of Theorem 1. We choose q. Then, $1/\alpha p < q < 1$. Let

$$X_{nk}^{(n)} = X_{nk} I(|X_{nk}| \leq n^{\alpha q}) + n^{\alpha q} I(|X_{nk}| > n^{\alpha q}) - n^{\alpha q} I(|X_{nk}| < -n^{\alpha q}),$$
$$Y_{nk} = (X_{nk} - n^{\alpha q}) I(|X_{nk}| > n^{\alpha q}) + (X_{nk} + n^{\alpha q}) I(|X_{nk}| < -n^{\alpha q}),$$

$$Y_{nk}^{+} = (X_{nk} - n^{\alpha q}) I(|X_{nk}| > n^{\alpha q}),$$

$$Y_{nk}^{-} = -(X_{nk} + n^{\alpha q}) I(|X_{nk}| < -n^{\alpha q}),$$

$$U_{nk} = \sum_{i=1}^{k} X_{ni}^{(n)}.$$

It is obvious that

$$\max_{k\le n}|S_k| \le \max_{k\le n}|U_{nk}| + \sum_{k=1}^{n}|Y_{nk}| \le \max_{k\le n}|U_{nk}| + \sum_{k=1}^{n}Y_{nk}^{+} + \sum_{k=1}^{n}Y_{nk}^{-}.$$

So we need only prove that

$$I =: \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P(\max_{k\le n}|U_{nk}| \ge \frac{\varepsilon}{2}n^{\alpha}) < \infty; \tag{6}$$

$$II =: \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\sum_{k=1}^{n}Y_{nk}^{+} \ge \frac{\varepsilon}{4}n^{\alpha}\right) < \infty; \tag{7}$$

$$III =: \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\sum_{k=1}^{n}Y_{nk}^{-} \ge \frac{\varepsilon}{4}n^{\alpha}\right) < \infty. \tag{8}$$

The proof of (7) and the proof of (8) are similar, so we only prove (7). Let

$$Z_{nk} = (X_{nk} - n^{\alpha q})I(n^{\alpha q} < X_{nk} \le n^{\alpha q} + n^{\alpha}) + n^{\alpha}I(X_{nk} > n^{\alpha q} + n^{\alpha}).$$

It is obvious that Z_{nk} is a monotone function of X_{nk} and

$$\{Y_{nk}^{+} \ne Z_{nk}\} \sqsubseteq \{X_{nk} > n^{\alpha}\}.$$

So

$$II \le \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)nP(|X_{nk}| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\left(\sum_{k=1}^{n}Z_{nk} > \frac{\varepsilon}{4}n^{\alpha}\right)$$

$$=: II_1 + II_2.$$

By (4) and

$$\sup_{n,k} P(|X_{nk}| > t) \le P(|X| > t)$$

for all $t \in R$, we have

$$II_1 \le C\sum_{n=1}^{\infty} n^{\alpha p-1}h(n)P(|X| > n^{\alpha}) \le E|X|^p h(|X|^{\frac{1}{\alpha}}) < \infty.$$

If $h(x) > 0$ is a slowly varying function when $x \to \infty$, then

$$\lim_{x\to\infty} x^{\delta} h(x) = \infty.$$

By (4), we have for all $0 < \delta < p$ that $E|X|^{p-\delta} < \infty$. Since $1/\alpha p < q < 1$, $\alpha pq > 1$. Choose δ such that $p - \delta - 1 > 0$, if $p > 1$ and $\alpha(p-\delta) > \alpha(p-\delta)q > 1$. Thus, when $p > 1$, $n \to \infty$, we have

$$\begin{aligned} n^{-\alpha}\sum_{k=1}^{n}|Y_{nk}| &\leq n^{-\alpha}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > n^{\alpha q}) \\ &\leq Cn^{-\alpha}\sum_{k=1}^{n} E|X|I(|X| > n^{\alpha q}) \\ &\leq Cn^{-\alpha(p-\delta)q+1-\alpha(1-q)} E|X|^{p-\delta} I(|X > n^{\alpha q}) \to 0. \end{aligned} \tag{9}$$

So

$$n^{-\alpha}\sum_{k=1}^{n} Z_{nk} \leq n^{-\alpha}\sum_{k=1}^{n}|Y_{nk}| \to 0.$$

When $p \leq 1$, $n \to \infty$, we have

$$\begin{aligned} n^{-\alpha}\sum_{k=1}^{n} Z_{nk} &\ll nP(|X| > n^{\alpha}) + n^{1-\alpha}E|X|I(|X| < 2n^{\alpha}) \\ &\leq n^{1-\alpha(p-\delta)}E|X|^{p-\delta} + n^{1-\alpha(p-\delta)}E|X|^{p-\delta}I(|X| < 2n^{\alpha}) \\ &\leq Cn^{1-\alpha(p-\delta)} \to 0. \end{aligned} \tag{10}$$

By Lemma 2, we have

$$
\begin{aligned}
II_2 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2}h(n)P\bigg(\sum_{k=1}^{n}(Z_{nk}-EZ_{nk})>\frac{\varepsilon}{8}n^{\alpha}\bigg)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}h(n)Var\bigg(\sum_{k=1}^{n}Z_{nk}\bigg)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}h(n)E(Z_{n1})^2\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}h(n)E(X-n^{\alpha q})^2 I(n^{\alpha q}<X\le n^{\alpha q}+n^{\alpha})\\
&+C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}h(n)n^{\alpha}I(X>n^{\alpha q}+n^{\alpha})\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-1}h(n)P(X>n^{\alpha})+C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}h(n)EX^2 I(|X|<2n^{\alpha})\\
&=: CII_1+CII_3.
\end{aligned}
$$

By (3), we have

$$
\begin{aligned}
II_3 &= \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}h(n)\sum_{j=1}^{n}EX^2 I(2(j-1)^{\alpha}<|X|\le 2j^{\alpha})\\
&= \sum_{j=1}^{\infty}\sum_{n=j}^{\infty} n^{\alpha p-1-2\alpha}h(n)EX^2 I(2(j-1)^{\alpha}<|X|\le 2j^{\alpha})\\
&\le C\sum_{j=1}^{\infty} j^{\alpha p-2\alpha}h(j)EX^2 I(2(j-1)^{\alpha}<|X|\le 2j^{\alpha})
\end{aligned}
$$

$$\leq C \sum_{j=1}^{\infty} j^{\alpha p} h(j) P(2(j-1)^{\alpha} < |X| \leq 2j^{\alpha})$$

$$\leq CE|X|^p h(|X|^{\frac{1}{\alpha}}) < \infty.$$

Thus, $II_2 < \infty$. In order to prove (6), we first prove that when $n \to \infty$,

$$n^{-\alpha} \max_{1\leq j\leq n} |EU_{nj}| \to 0.$$

When $\alpha \leq 1$, since $\alpha p > 1$, $p > 1$. Notice that $EX = 0$. By (9), when $n \to \infty$, then

$$n^{-\alpha} \max_{1\leq j\leq n} |EU_{nj}| = n^{-\alpha} \max_{1\leq j\leq n} \left| \sum_{k=1}^{j} EY_{nk} \right| \leq n^{-\alpha} \sum_{k=1}^{n} E|Y_{nk}| \to 0.$$

When $\alpha > 1$, $p > 1$, then

$$n^{-\alpha} \max_{1\leq j\leq n} |EU_{nj}| \leq n^{-\alpha} \sum_{k=1}^{n} E|X_{nk}| \leq n^{1-\alpha} E|X| \leq Cn^{1-\alpha} \to 0.$$

When $\alpha > 1$, $p \leq 1$, then

$$n^{-\alpha} \max_{1\leq j\leq n} |EU_{nj}| \leq n^{1-\alpha} E|X|$$

$$\leq nP(|X| > n^{\alpha}) + n^{1-\alpha} E|X| I(|X| < n^{\alpha})$$

$$\leq n^{1-\alpha(p-\delta)} E|X|^{p-\delta} + n^{1-\alpha(p-\delta)} E|X|^{p-\delta} I(|X| < n^{\alpha})$$

$$\leq Cn^{1-\alpha(p-\delta)} \to 0.$$

We now complete the proof of (11). Choose $0 < \delta < p$, such that $2-p+\delta > 0$, $q\delta < (2-p)(1-q)$. By (11) and Lemma 2, we have that

$$\begin{aligned}
I_2 &\le C \sum_{n=1}^{\infty} n^{\alpha p-2} h(n) P\left(\max_{1\le j\le n} \left| \sum_{k=1}^{j} (X_{nk}^{(n)} - EX_{nk}^{(n)}) \right| > \frac{\varepsilon}{8} n^{\alpha} \right) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} h(n) E \max_{1\le j\le n} \left| \sum_{k=1}^{j} (X_{nk}^{(n)} - EX_{nk}^{(n)}) \right|^2 \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} h(n) \log^2 n (EX_{n1}^{(n)})^2 \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} h(n) n^{\alpha q(2-p+\delta)} \log^2 n E|X|^{p-\delta} \\
&\le C \sum_{n=1}^{\infty} n^{-1-\alpha(2-p)(1-q)+\alpha q\delta} h(n) \log^2 n \\
&< \infty.
\end{aligned}$$

We now complete the proof of Theorem 1.

Remark 1. Let $h(x) \equiv 1$, $\alpha p = 2$. Notice that pairwise NQD is more general than ND. Using Theorem 1, we can get Theorem 3.1 (iii) in Taylor [2].

Corollary 1. Let $h(x) \equiv 1$, $\alpha p = 2$. By $E|X|^p < \infty$, when $n \to \infty$, we have

$$\frac{S_n}{n^{2/p}} \to 0 \quad \text{a.s.}$$

This is because a negative quadrant dependent (NQD) sequence is more general than a linear negative quadrant dependence (LNQD) sequence or negatively associated (NA) sequence. Using Theorem 1, we have the following two Corollaries.

Corollary 2. Let $\{X_{nk},\ 1 \le k \le n,\ n \ge 1\}$ be an array of rowwise LNQD random variables with $EX_{nk} = 0$ for each n and k. Let

$$S_n = \sum_{k=1}^{n} X_{nk}.$$

$$\sup_{n,k} P(|X_{nk}| > t) \le P(|X| > t)$$

for all $t \in R$.

$$E|X|^p h(|X|^{\frac{1}{\alpha}}) < \infty, \tag{12}$$

where $\alpha p > 1$, $0 < p < 2$, $h(x) > 0$ is a slowly varying function when $x \to \infty$. When $\alpha \le 1$, $EX = 0$. Then

$$\text{for all } \varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{p\alpha-2} h(n) P(\max_{1\le j\le n} |S_j| > \varepsilon n^{\alpha}) < \infty. \tag{13}$$

Corollary 3. Let $\{X_{nk},\ 1 \le k \le n,\ n \ge 1\}$ be an array of rowwise NA random variables with $EX_{nk} = 0$ for each n and k. Let

$$S_n = \sum_{k=1}^{n} X_{nk}.$$

$$\sup_{n,k} P(|X_{nk}| > t) \le P(|X| > t)$$

for all $t \in R$.

$$E|X|^p h(|X|^{\frac{1}{\alpha}}) < \infty, \tag{14}$$

where $\alpha p > 1$, $0 < p < 2$, $h(x) > 0$ is a slowly varying function when $x \to \infty$. When $\alpha \le 1$, $EX = 0$. Then,

$$\text{for all } \varepsilon > 0, \sum_{n=1}^{\infty} n^{p\alpha-2} h(n) P(\max_{1\le j\le n} |S_j| > \varepsilon n^{\alpha}) < \infty. \tag{15}$$

For the definitions of LNQD sequence and NA sequence see the following.

Definition 3 (Zhang [6]). A random variable sequence $\{X_k, k \geq 1\}$ is said to be LNQD, if for any disjoint A, B and a positive constant sequence $\{r_j, j \geq 1\}$, then

$$\sum_{i \in A} r_i X_i \quad \text{and} \quad \sum_{j \in B} r_j X_j$$

are NQD.

Definition 4 (Joag [4]). A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \ldots, n\}$, we have

$$Cov(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

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