ON THE RATIO VECTORS OF CHEBYSHEV AND EQUISPACED POLYNOMIALS

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Abstract. Every real polynomial with all real, distinct zeros has a corresponding ratio vector that measures the distribution of critical numbers relative to the zeros. We prove several results on the ratio vectors of Chebyshev polynomials, as well as polynomials with equally spaced zeros, and contrast the behaviors of these families of polynomials.

Introduction. Real polynomials of degree n with n distinct real zeros form a beautiful class of functions. By Rolle's Theorem, between every pair of zeros lies exactly one critical number, and the polynomial will oscillate between relative minima and relative maxima as we proceed from one critical number to the next. Important examples of such functions abound – the classical orthogonal polynomials are particularly notable.

In [2], P. Andrews used the Polynomial Root-Dragging Theorem [1] to prove that for a degree n polynomial $p(x)$ with distinct real zeros $r_1 < r_2 < \cdots < r_n$ and critical numbers $c_1 < c_2 < \cdots < c_{n-1}$, then

$$\frac{1}{n-i+1} < \frac{c_i - r_i}{r_{i+1} - r_i} < \frac{i}{i+1}, \quad i = 1, \ldots, n.$$

(This result was also proved by G. Peyser in 1967, almost 30 years prior to Andrews' work, using a different argument. While both were published in the *Monthly,* it appears that Peyser's work is less well-known.) An immediate consequence of Andrews' result is that no polynomial (with distinct real zeros) of degree greater than 2 can have all of its critical numbers lying exactly halfway between consecutive roots; in particular, the leftmost critical number must lie less than halfway between the first two roots, and the rightmost critical number must lie more than halfway across the interval formed by the last two roots.

Letting $\sigma_i = \frac{c_i - r_i}{r_{i+1} - r_i}$ and $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$, σ_i is called the i^{th} *ratio* of p, and σ is the *ratio vector* of the polynomial p. Andrews showed that for every cubic polynomial, $\sigma_1 < \sigma_2$; in [3], A. Horwitz proved that for every quartic polynomial, $\sigma_1 < \sigma_2 < \sigma_3$, and termed such polynomials as having a "monotonic ratio vector." Horwitz went on to show that every degree n polynomial with n *equally spaced* real zeros (an "equispaced" polynomial) has a monotonic ratio vector, but that for every degree $n \geq 5$, there exist polynomials with non-monotonic ratio vectors.

In this paper, we consider the famous Chebyshev polynomials and compare them to polynomials with equally spaced zeros. In particular, we show that the ratio vector of every Chebyshev polynomial is monotonic (Theorem 1), and also address the distribution of the ratios as n increases without bound (Theorems 2, 3). Moreover, we take a closer look at the ratio vector of equispaced polynomials, and prove similar results to Theorems 2 and 3 on Chebyshev polynomials (Theorems 4, 5). While many results on the distribution of zeros of certain families of polynomials are known, the propositions presented here focus on the distribution of critical numbers, and do so *relative to* the zeros of the polynomial under consideration. We'll see that the critical numbers of Chebyshev polynomials stay very near the midpoint of consecutive roots, while for polynomials with equally spaced zeros, certain critical numbers tend toward the outer roots of the interval in which they lie.

Chebyshev Polynomials. Chebyshev polynomials are a classic example of orthogonal polynomials. In particular, they are a special case of ultraspherical Jacobi polynomials, and are orthogonal with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx$. The roots of Chebyshev polynomials find important applications in numerical analysis, since using the roots as interpolatory nodes helps minimize the oscillations of interpolating polynomials [4].

The degree n Chebyshev polynomial is defined by $T_n(x) = \cos(n \arccos(x))$. It follows that the roots, r_j, and the critical numbers, c_j, of T_n (in increasing order) are

$$r_j = -\cos\left(\frac{(2j-1)\pi}{2n}\right), j = 1, \dots, n, \tag{1}$$

and

$$c_j = -\cos\left(\frac{j\pi}{n}\right), j = 1, \dots, n-1. \tag{2}$$

Knowing both the roots and critical numbers explicitly, these formulas allow us to easily consider properties of the ratio vector, σ. One is that every Chebyshev polynomial has a monotonic ratio vector.

Theorem 1. For every $n \geq 3$, the Chebyshev polynomial $T_n(x)$ has a monotonically increasing ratio vector.

Proof. Let $T_n(x)$ be a Chebyshev polynomial of fixed degree $n \geq 3$ and $j \in \{1, \ldots, n-1\}$. Using (1) and (2) and angle sum trig identities, we observe that

$$\sigma_j = \frac{c_j - r_j}{r_{j+1} - r_j} = \frac{\cos\left(\frac{2j\pi}{2n}\right) - \cos\left(\frac{(2j-1)\pi}{2n}\right)}{\cos\left(\frac{(2j+1)\pi}{2n}\right) - \cos\left(\frac{(2j-1)\pi}{2n}\right)} =$$

$$\frac{\cos\left(\frac{2j\pi}{2n}\right) - \cos\left(\frac{2j\pi}{2n}\right)\cos\left(\frac{-\pi}{2n}\right) + \sin\left(\frac{2j\pi}{2n}\right)\sin\left(\frac{-\pi}{2n}\right)}{\cos\left(\frac{2j\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right) - \sin\left(\frac{2j\pi}{2n}\right)\sin\left(\frac{\pi}{2n}\right) - \cos\left(\frac{2j\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right) + \sin\left(\frac{2j\pi}{2n}\right)\sin\left(\frac{-\pi}{2n}\right)}.$$

Some simplification reveals that

$$\sigma_j = -\frac{1}{2}\cot\left(\frac{j\pi}{n}\right)\left[\frac{1-\cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)}\right] + \frac{1}{2}. \tag{3}$$

Since $\cot\left(\frac{j\pi}{n}\right)$ is a decreasing function of j, we see that $-\frac{1}{2}\cot\left(\frac{j\pi}{n}\right)$ is increasing in j. Therefore, σ_j will increase as j increases, and the ratio vector of any Chebyshev polynomial is monotonically increasing.

Moreover, (3) tells us that the ratios between positive roots of a Chebyshev polynomial increase at an increasing rate, in accordance with the concavity of the negative cotangent function.

At this point, we have two prominent examples of families of polynomials with monotone ratio vectors: the aforementioned polynomials with equally spaced zeros and the Chebyshev polynomials. Numerical testing of many examples of ultraspherical Jacobi polynomials leads us to conjecture that every ultraspherical Jacobi polynomial has a monotone ratio vector, lacking the simplicity of closed forms for the roots and critical numbers of the Chebyshev polynomials; it is clear that other methods will be required to prove this hypothesis.

However, there is much more we can say about Chebyshev polynomials, as well as those with equally spaced zeros. Numerical investigation of ratio vectors of these families reveals some interesting patterns: in both types of functions, the ratios in intervals between consecutive roots near the origin appear to approach $\frac{1}{2}$, while the ratios nearest the outer root of the respective polynomials behave very differently. In the next sections, we explore these phenomena in more detail.

Select Ratios of Chebyshev Polynomials. We first consider what we will call the "inner" ratios of a Chebyshev polynomial. In particular, for an odd Chebyshev polynomial of degree $2n+1$, it is natural to examine σ_{n+1}, the first "positive" ratio between roots $r_{n+1}=0$ and r_{n+2}. We'll go on to consider σ_{n+j} for a given fixed j and, in each case, allow the degree of the polynomial to increase. It turns out that all such ratios decrease monotonically to $\frac{1}{2}$ as n increases without bound. We'll then examine the "outer" ratios, σ_{2n-j}, $j=0,1,\ldots$, looking at those among the outermost roots and find an intriguing pattern there as well. Similar results hold for the even case.

We briefly note that for an odd polynomial of degree $2n+1$, it is straightforward to verify that $\sigma_{n-j}+\sigma_{n+j+1}=1$ $(j=1,\ldots,n-1)$, due to symmetry. For an even polynomial of degree $2n$, $\sigma_n=1/2$, and $\sigma_{n-j}+\sigma_{n+j}=1$ $(j=1,\ldots,n-1)$. As such, it always suffices to consider the "positive" ratios between positive roots.

Theorem 2. Let $T_{2n+1}(x)$ be a degree $2n+1$ Chebyshev polynomial. Then for each $j\in\mathbb{N}$,

$$\lim_{n\to\infty}\sigma_{n+j}=\frac{1}{2}.$$

Proof. Let $T_{2n+1}(x)$ be an odd Chebyshev polynomial of degree $2n+1$ and j be a natural number. Recalling (3), and replacing n by $2n+1$ and j by $n+j$, we have that

$$\sigma_{n+j}=-\frac{1}{2}\cot\left(\frac{(n+j)\pi}{2n+1}\right)\left[\frac{1-\cos\left(\frac{\pi}{4n+2}\right)}{\sin\left(\frac{\pi}{4n+2}\right)}\right]+\frac{1}{2}. \tag{4}$$

Note that (for sufficiently large n)

$$-\frac{1}{2}\cot\left(\frac{(n+j)\pi}{2n+1}\right)$$

is a positive number that decreases to zero as $n \to \infty$. It is straightforward to verify that

$$\frac{1-\cos\left(\frac{\pi}{4n+2}\right)}{\sin\left(\frac{\pi}{4n+2}\right)}$$

is a positive decreasing function of n, so therefore, the nonconstant portion of (4) decreases to zero.

Hence, σ_{n+j} is a decreasing function of n and $\lim_{n\to\infty}\sigma_{n+j} = \frac{1}{2}$ for each fixed $j > 0$.

A similar argument proves that the inner ratios of even Chebyshev polynomials tend to 1/2. In that case, we note that σ_n, the "center" ratio, is precisely 1/2 for all n, because that particular critical number is always zero, centered between the first positive and first negative root.

We next look at the "outer" ratios of $T_n(x)$ and find a pattern in their behavior. Here it is no longer necessary to distinguish between the odd and even cases, and it again suffices to consider the ratios to the right of the origin.

Theorem 3. Let j be a natural number and $T_n(x)$ a degree n Chebyshev polynomial. Consider the j^{th} ratio from the rightmost root of T_n, σ_{n-j}. Then

$$\lim_{n\to\infty}\sigma_{n-j} = \frac{4j+1}{8j}.$$

Proof. Let j be a natural number with n sufficiently large. Using (1) and (2), we substitute c_{n-j}, r_{n-j}, and r_{n-j+1} appropriately, and take the limit as $n \to \infty$. Hence,

$$\begin{aligned}
\lim_{n\to\infty}\sigma_{n-j} &= \lim_{n\to\infty}\frac{c_{n-j}-r_{n-j}}{r_{n-j+1}-r_{n-j}} \\
&= \lim_{n\to\infty}\frac{-\cos\left(\frac{2(n-j)\pi}{2n}\right)+\cos\left(\frac{(2(n-j)-1)\pi}{2n}\right)}{-\cos\left(\frac{(2(n-j+1)-1)\pi}{2n}\right)+\cos\left(\frac{(2(n-j)-1)\pi}{2n}\right)} \\
&= \lim_{n\to\infty}\frac{\cos\left(\pi-\frac{2j\pi}{2n}\right)-\cos\left(\pi-\frac{(2j+1)\pi}{2n}\right)}{\cos\left(\pi-\frac{(2j-1)\pi}{2n}\right)-\cos\left(\pi-\frac{(2j+1)\pi}{2n}\right)}. \qquad (5)
\end{aligned}$$

Following two applications of L'Hôpital's Rule and some simplification, we find that

$$\lim_{n\to\infty} \sigma_{n-j}$$

$$= \lim_{n\to\infty} \frac{-\cos\left(\pi - \frac{2j\pi}{2n}\right)[2j]^2 + \cos\left(\pi - \frac{(2j+1)\pi}{2n}\right)[2j+1]^2}{-\cos\left(\pi - \frac{(2j-1)\pi}{2n}\right)[2j-1]^2 + \cos\left(\pi - \frac{(2j+1)\pi}{2n}\right)[2j+1]^2}.$$

As n approaches ∞, each of the expressions involving the cosine function tend to -1, so it follows that

$$\lim_{n\to\infty} \sigma_{n-j} = \frac{(2j)^2 - (2j+1)^2}{(2j-1)^2 - (2j+1)^2} = \frac{4j+1}{8j}. \tag{6}$$

This result shows that the overall distribution of critical numbers in Chebyshev polynomials of large degree is very "centered": the leftmost ratios can go as low as 3/8, the center ratios all approach 1/2 as n gets large, and the outermost ratios only end up as large as 5/8, with a strict pattern of increase and decrease as one moves in from the extremes. In fact, 5/8 is an upper bound on the rightmost ratio, and therefore, on all ratios of a Chebyshev polynomial. It is not difficult to prove this bound by setting $j = 1$ in (5).

This near centering of the critical numbers of Chebyshev polynomials stands in contrast to the behavior of odd and even polynomials with equally spaced zeros, where the ratios eventually range from arbitrarily close to zero to as near to 1 as we like.

Select Ratios of Polynomials with Equally Spaced Zeros. In [5], K. MacLean proved that for odd polynomials with equally spaced zeros, the outermost critical number approaches the outermost root. That is, for the odd polynomials with roots at $0, \pm 1, \ldots, \pm n$, and critical numbers $\pm c_0, \ldots, \pm c_{n-1}$, $\lim_{n\to\infty}(n - c_{n-1}) = 0$. Said differently, this says that the outermost ratio of such an "equispaced" polynomial approaches 1 as the degree of the polynomial is allowed to increase without bound.

In what follows, we first generalize the argument in [5] to show that for any fixed j, the j outermost positive ratios all tend to 1 as the degree of the polynomial increases without bound. In addition, we prove a result regarding the positive ratios near the origin, showing that, like the Chebyshev polynomials, these all decrease to $1/2$ as n increases.

For the rest of the paper we modify our notation slightly. For an odd polynomial we let

$$p(x) = x \prod_{i=1}^{n} (x^2 - r_i^2),$$

where $0 < r_1 < \cdots < r_n$ are the distinct real, non-negative roots of p. When convenient, we will call $r_0 = 0$. In addition, we denote the critical numbers by $\pm c_0, \ldots, \pm c_{n-1}$, where the positive critical numbers increase with the index n. As noted earlier, due to symmetry about the origin, we can simply consider "positive" ratios

$$\sigma_i = \frac{c_i - r_i}{r_{i+1} - r_i},$$

$i = 0, \ldots, n-1$. If needed, we refer to the corresponding negative ratio, σ_{-i}, given by

$$\sigma_{-i} = \frac{-c_i + r_{i+1}}{-r_i + r_{i+1}},$$

and again remark that $\sigma_{-i} + \sigma_i = 1$.

Theorem 4. Let

$$p(x) = x \prod_{i=1}^{n} (x^2 - i^2)$$

be an odd polynomial with equally spaced zeros, and choose a natural number j. Then $\lim_{n\to\infty} \sigma_{n-j} = 1$.

Proof. Let $j \in \mathbb{N}$ and n be sufficiently large. Consider the critical point c_{n-j} between the roots $n-j$ and $n-j+1$. For some ϵ_n such that $0 < \epsilon_n < 1$, we have that $c_{n-j} = n - j + 1 - \epsilon_n$. The corresponding ratio, σ_{n-j}, is given by $1 - \epsilon_n$, so we simply need to show that $\epsilon_n \to 0$ as $n \to \infty$.

Observe that

$$\frac{p'(x)}{p(x)} = \frac{1}{x+n} + \cdots + \frac{1}{x+1} + \frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-n} = \sum_{i=-n}^{n} \frac{1}{x-i}.$$

Since $c_{n-j} = n - j + 1 - \epsilon_n$ is a critical point of $p(x)$, we have

$$\frac{p'(n-j+1-\epsilon_n)}{p(n-j+1-\epsilon_n)} = 0 = \sum_{i=-n}^{n} \frac{1}{n-j+1-\epsilon_n - i}. \tag{7}$$

Rewriting the righthand equation in (7) and breaking it into two separate sums, it follows that

$$\sum_{i=0}^{j-1} \frac{1}{i+\epsilon_n} = \frac{1}{2n-j+1-\epsilon_n} + \cdots + \frac{1}{1-\epsilon_n}.$$

Since $\epsilon_n > 0$ and $\frac{1}{i+\epsilon_n} < \frac{1}{\epsilon_n}$ for all $i \geq 1$,

$$\sum_{i=0}^{j-1} \frac{1}{\epsilon_n} > \sum_{i=0}^{j-1} \frac{1}{i+\epsilon_n} = \frac{1}{2n-j+1-\epsilon_n} + \cdots + \frac{1}{1-\epsilon_n} > \frac{1}{2n-j+1} + \cdots + \frac{1}{1}. \tag{8}$$

Hence, we find that

$$\frac{j}{\epsilon_n} > \sum_{i=1}^{2n-j+1} \frac{1}{i}.$$

Since

$$\sum_{i=1}^{2n-j+1} \frac{1}{i}$$

diverges as $n \to \infty$, we conclude that ϵ_n goes to 0. Thus, $\lim_{n\to\infty} \sigma_{n-j} = 1$.

Next, as with the Chebyshev polynomials, we examine the positive ratios near the origin and consider the behavior of these values as the degree of an odd polynomial with equally spaced zeros increases without bound. Specifically, we will consider the j^{th} ratio to the right of the origin, denoting this by σ_j. Note particularly that, due to the indexing of $r_0 = 0$, there is a "zeroth" ratio, σ_0, that measures the ratio of the critical number between $r_0 = 0$ and $r_1 = 1$.

Theorem 5. Let

$$E_n(x) = x \prod_{i=1}^{n} (x^2 - i^2)$$

be an odd polynomial of degree $2n+1$ with equally spaced zeros, and j be a fixed nonnegative integer. If σ_j denotes the j^{th} positive ratio of $E_n(x)$, then

$$\lim_{n\to\infty} \sigma_j = \frac{1}{2}.$$

Proof. Since $E_n(x)$ has roots located at integer values, we first see that

$$\lim_{n\to\infty} \sigma_j = \lim_{n\to\infty} \frac{c_j - r_j}{r_{j+1} - r_j} = \lim_{n\to\infty} \frac{c_j - j}{j+1-j} = \lim_{n\to\infty} c_j - j. \tag{9}$$

Thus, our problem of finding the limit of the j^{th} positive ratio has been narrowed to the problem of finding the limit of the j^{th} positive critical number as $n \to \infty$. For clarity, we will denote this critical number by $c_{j,n}$, and remark explicitly that $c_{j,n} \in [j, j+1]$.

Let $n \geq j+1$ be a natural number. We first show that $c_{j,n} \in [j + \frac{1}{2}, j+1)$. The upper bound is obvious. For the lower bound, assume to the contrary that $c_{j,n} < j + \frac{1}{2}$. Hence, $c_{j,n} = j + t_n$, where $0 < t_n < \frac{1}{2}$. It follows that

$$\sigma_j = c_{j,n} - j = t_n < \frac{1}{2}.$$

Since E_n is odd, the corresponding negative ratio σ_{-j} is $1 - t_n > 1/2$. This contradicts the known fact that polynomials with equally spaced zeros have monotonic ratio vectors. Thus, $c_{j,n}$ is bounded below by $j + \frac{1}{2}$.

Next, we proceed to show that as the degree of $E_n(x)$ increases, where $E_n(x)$ remains an odd polynomial with equally spaced zeros, $c_{j,n}$ decreases monotonically. We consider

$$\frac{E_n'(x)}{E_n(x)} = \sum_{i=1}^{n} \frac{2x}{x^2 - i^2} + \frac{1}{x} \tag{10}$$

and note that evaluating (10) at $c_{j,n}$ gives

$$\frac{E_n'(c_{j,n})}{E_n(c_{j,n})} = \sum_{i=1}^{n} \frac{2c_{j,n}}{c_{j,n}^2 - i^2} + \frac{1}{c_{j,n}} = 0. \tag{11}$$

Now consider $E_{n+1}(x)$, a polynomial of degree $2n+3$, where $E_{n+1}(x)$ has additional roots at $\pm(n+1)$. Evaluating at $c_{j,n}$, we find that, by (11),

$$\begin{aligned} \frac{E_{n+1}'(c_{j,n})}{E_{n+1}(c_{j,n})} &= \sum_{i=1}^{n+1} \frac{2c_{j,n}}{c_{j,n}^2 - i^2} + \frac{1}{c_{j,n}} \\ &= \sum_{i=1}^{n} \frac{2c_{j,n}}{c_{j,n}^2 - i^2} + \frac{1}{c_{j,n}} + \frac{2c_{j,n}^2}{c_{j,n}^2 - (n+1)^2} \\ &= \frac{2c_{j,n}}{c_{j,n}^2 - (n+1)^2}. \end{aligned}$$

Since $0 \le j < c_{j,n} < j+1 \le n < n+1$, it follows that $c_{j,n}^2 < (n+1)^2$, so

$$\frac{E_{n+1}'(c_{j,n})}{E_{n+1}(c_{j,n})} < 0. \tag{12}$$

The j^{th} positive critical number of $E_{n+1}(x)$ is $c_{j,n+1}$, so we know that

$$\frac{E_{n+1}'(c_{j,n+1})}{E_{n+1}(c_{j,n+1})} = 0. \tag{13}$$

Since $\frac{E'_{n+1}(x)}{E_{n+1}(x)}$ is a strictly decreasing function between its asymptotes, (12) and (13) together imply that $c_{j,n+1} < c_{j,n}$. Hence, as we increase the degree of E_n, the critical point between j and $j+1$ will decrease monotonically. Having already proved $c_{j,n}$ is bounded below by $j+1/2$, it follows that for each integer $j \geq 0$, there exists some $L_j \in [j+1/2, j+1)$ such that

$$\lim_{n\to\infty} c_{j,n} = L_j.$$

We conclude by showing that L_j is in fact $j+1/2$. Multiplying (11) through by $c_{j,n}$ and subtracting 1, we find that

$$\sum_{i=1}^{n} \frac{2c_{j,n}^2}{c_{j,n}^2 - i^2} = -1. \tag{14}$$

Letting $n \to \infty$ in (14), we observe that

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{2c_{j,n}^2}{c_{j,n}^2 - i^2} \right) = -1. \tag{15}$$

It is now natural to consider the function

$$f(L) = \sum_{i=1}^{\infty} \frac{2L^2}{L^2 - i^2}.$$

Using the calculus of residues as applied to the summation of real series [7], it is possible to show that

$$f(L) = \sum_{i=1}^{\infty} \frac{2L^2}{L^2 - i^2} = \pi \cot(\pi L) - 1. \tag{16}$$

(*The authors thank Paul Fishback for suggesting this elegant argument.*)

This shows that f is continuous on $(j, j+1)$ for all nonnegative integers j and, along with (15), allows us to conclude that

$$\pi \cot(\pi L_j) - 1 = \sum_{i=1}^{\infty} \frac{2L_j^2}{L_j^2 - i^2} = -1. \tag{17}$$

Recalling that $L_j \in (j, j+1)$, it follows immediately that $L_j = \frac{2j+1}{2} = j + \frac{1}{2}$ for each $j \geq 0$. Thus, $c_{j,n} \to L_j = j + \frac{1}{2}$, and therefore, each of the first j positive ratios tend to $\frac{1}{2}$ as we increase the degree of the polynomial without bound.

Results similar to Theorems 4 and 5 hold in the case that E_n is an even polynomial with equally spaced zeros.

Conclusion. These results on the ratio vectors of Chebyshev and equispaced polynomials provide some insight into the very different behavior of these functions, as shown in Figure 1. Because the ratios of Chebyshev polynomials are always close to 1/2, it seems that the function is naturally minimizing the amount of room it has available to grow until it reaches a turning point. This is also seen in the "equioscillatory" behavior of the polynomial. Equispaced polynomials, on the other hand, push the outer critical numbers near the outermost roots, in some sense allowing more space for the function to increase or decrease until having to turn. Of course, these phenomena are due directly to the (very different) distribution of zeros, from which the respective distributions of critical numbers follows.

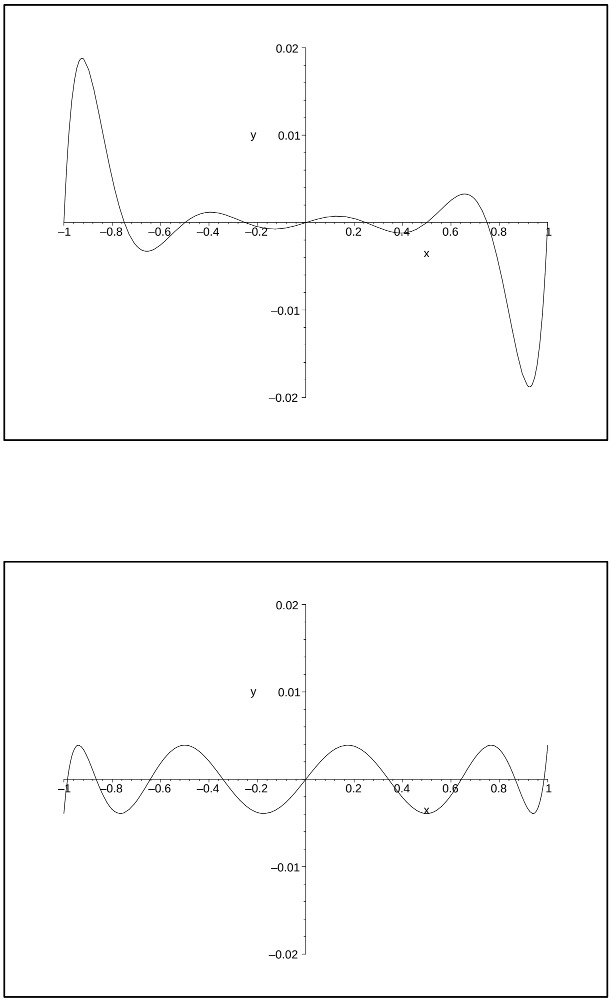

Figure 1: Degree 9 equispaced and Chebyshev polynomials. The equispaced polynomial is scaled to have all its zeros in $[-1, 1]$; the Chebyshev polynomial is scaled to be monic.

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