

BOOTSTRAPPING OUR WAY TO THE PRODUCT RULE

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Abstract. In this paper, a technique is introduced for proving the product rule for the differentiation of the product of two functions in an undergraduate real analysis class. The general form of the product rule is proved using the sum rule and constant multiple rule as well as a special case of the product rule. This special case can be proved quite easily using only the definition of the derivative and the fact that a differentiable function is continuous.

An undergraduate real analysis course should provide students with a bridge between the intuitive notions of beginning calculus and more general concepts in advanced mathematics courses. While it is essential that students see and do rigorous proofs, they should also see the connectivity and consistency of the underlying concepts.

This is exemplified in Belding and Mitchell's approach to the development of the elementary rules of differentiation in their real analysis textbook. For example, instead of using a formal limit proof of the quotient rule for differentiation, they prove it by first presenting a special case of the quotient rule [1].

If f is differentiable at x and $f(x) \neq 0$, then $\frac{d}{dx} \left[\frac{1}{f(x)} \right] = -\frac{f'(x)}{[f(x)]^2}$.

Proving this is an easy exercise for students that uses only the definition of the derivative and the fact that a differentiable function is continuous. Assuming the product rule has already been proved, the general quotient rule then follows easily (again as a good student exercise) by writing the quotient

$$\frac{f(x)}{g(x)} \text{ as } f(x) \cdot \frac{1}{g(x)}$$

and using the product rule and the special case of the quotient rule [1].

This approach could be thought of as "bootstrapping," since a special case of a formula is used to prove the more general formula.

In a similar vein, we present a "bootstrapping" approach to the development of the product rule. First, a special case is introduced that can easily be proved as an exercise by students and then, by using this special case, the more general product rule is established.

Let f be differentiable at x . The derivative formula

$$\frac{d}{dx}([f(x)]^2) = \frac{d}{dx}[f(x) \cdot f(x)] = 2f(x)f'(x)$$

is a special case of the product rule. Let's call it the Square Rule. This formula can be proved directly using the definition of the derivative without recourse to the product rule. In order to motivate the proof, the diagram in Figure 1 can be presented to show pictorially the relation between the various components used in the proof.

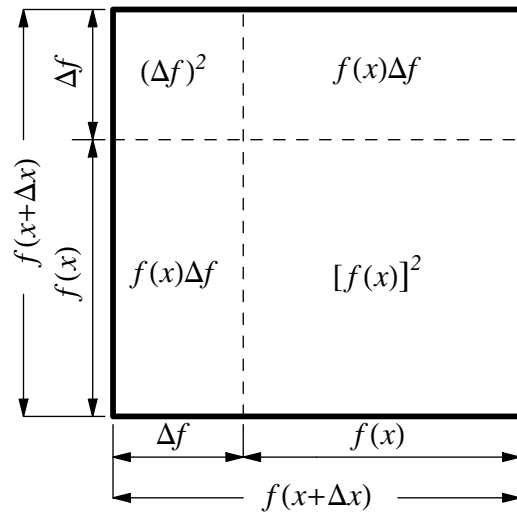


Figure 1.

From the diagram it can be seen that the area of the large square, $[f(x + \Delta x)]^2$, is equal to the sum of the areas of the included squares and rectangles. This gives rise to the equation

$$[f(x + \Delta x)]^2 = [f(x)]^2 + (\Delta f)^2 + 2f(x)\Delta f,$$

which can be rewritten as

$$[f(x + \Delta x)]^2 - [f(x)]^2 = (\Delta f)^2 + 2f(x)\Delta f.$$

Dividing both sides by Δx , we have

$$\frac{[f(x + \Delta x)]^2 - [f(x)]^2}{\Delta x} = \frac{\Delta f}{\Delta x} \Delta f + 2f(x) \frac{\Delta f}{\Delta x}.$$

Taking the limit of both sides of this equation as $\Delta x \rightarrow 0$ and noting that $\Delta f = f(x + \Delta x) - f(x)$, we now have

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x)]^2 - [f(x)]^2}{\Delta x} \\ = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} \cdot \lim_{\Delta x \rightarrow 0} [f(x + \Delta x) - f(x)] + 2f(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}. \end{aligned}$$

The left side of this equation is the derivative of $[f(x)]^2$ and

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$$

is equal to $f'(x)$. In addition, the role of continuity of f comes into play since continuity implies $\lim_{\Delta x \rightarrow 0} [f(x + \Delta x) - f(x)] = 0$. We conclude that

$$\frac{d}{dx}([f(x)]^2) = 2f(x)f'(x)$$

and the Square Rule is established. If a more formal approach is desired, the proof of the Square Rule could be developed more succinctly.

$$\begin{aligned} \frac{d}{dx}([f(x)]^2) &= \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x)]^2 - [f(x)]^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x) - f(x)][f(x + \Delta x) + f(x)]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x) - f(x)]}{\Delta x} [f(x + \Delta x) + f(x)] \\ &= f'(x)[2f(x)] = 2f(x)f'(x). \end{aligned}$$

Here again, continuity of f is necessary in order to conclude that $\lim_{\Delta x \rightarrow 0} ([f(x + \Delta x) + f(x)]) = 2f(x)$.

Now that this formula has been established, it can be used to prove the more general product rule. Let f and g be two differentiable functions. We compute

$$\frac{d}{dx}([f(x) + g(x)]^2)$$

two different ways. First we apply the Square Rule and the sum rule to obtain

$$\begin{aligned} \frac{d}{dx}([f(x) + g(x)]^2) &= 2[f(x) + g(x)][f(x) + g(x)]' & (1) \\ &= 2[f(x) + g(x)][f'(x) + g'(x)] \\ &= 2f(x)f'(x) + 2f(x)g'(x) + 2g(x)f'(x) + 2g(x)g'(x). \end{aligned}$$

Secondly, by expanding $[f(x) + g(x)]^2$ first and then taking the term-by-term derivative using the constant multiple rule, the sum rule and the square rule twice, we obtain

$$\begin{aligned} \frac{d}{dx}([f(x) + g(x)]^2) &= \frac{d}{dx}([f(x)]^2 + 2f(x)g(x) + [g(x)]^2) & (2) \\ &= 2f(x)f'(x) + 2\frac{d}{dx}[f(x)g(x)] + 2g(x)g'(x). \end{aligned}$$

Then, by equating the right-hand expressions in (1) and (2) and cancelling common terms, we arrive at the product rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

This development of the product rule would make a highly instructive set of homework or in-class exercises for the student, bringing into play the relation between the definitions of limits, derivatives, and continuity.

Reference

1. D. Belding and K. Mitchell, *Foundations of Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

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