ON FACTORIZATIONS OF ESCALATOR NUMBERS

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Abstract. Questions concerning the factorizations of escalator numbers are considered. Specifically, we determine exactly when escalator numbers are nontrivial integral powers of rational numbers. We further bound the number of primes and thus the size of the largest prime dividing the numerator and denominator of an escalator number in terms of its position in an escalator number sequence.

1. Introduction. A sequence $a_1, a_2, \ldots$ of rational numbers is called an *escalator sequence* if, for each $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i.$$

An *escalator number* is a partial sum of an escalator sequence

$$A_n = \sum_{i=1}^{n} a_i,$$

with $n \geq 2$. The sequence $A_2, A_3, \ldots$ of escalator numbers is an *escalator number sequence.* For basic results on escalator numbers and sequences, see works by Pizá [6] and by Grundman [3,4].

Note that for $n \geq 2$, $A_{n-1} + a_n = a_n A_{n-1}$ and therefore, assuming that $A_{n-1} \neq 1$,

$$a_n = \frac{A_{n-1}}{A_{n-1} - 1}.$$

Hence,

$$A_n = a_n A_{n-1} = \frac{A_{n-1}^2}{A_{n-1} - 1}. \tag{1}$$

Since $A_1 = a_1$, it follows that if $a_1 \neq 1$, then both a_n and A_n are uniquely determined in terms of a_1, called the *base* of the resulting escalator sequence. Equation (1) also demonstrates that for each n, $A_n \neq 1$ since otherwise $x = A_{n-1}$ would be a rational solution to $x^2/(x-1) = 1$, which is impossible.

To fix notation, given a base $a_1 \neq 1$, for each $n \in \mathbb{Z}^+$, let $u_n \in \mathbb{Z}$ and $v_n \in \mathbb{Z}^+$ with $\gcd(u_n, v_n) = 1$ such that

$$A_n = \frac{u_n}{v_n}.$$

Using (1),

$$A_{n+1} = \frac{A_n^2}{A_n - 1} = \frac{u_n^2}{v_n(u_n - v_n)}, \tag{2}$$

with $\gcd(u_n^2, v_n(u_n - v_n)) = 1$. Therefore,

$$u_{n+1} = \pm u_n^2 \quad \text{and} \quad v_{n+1} = v_n|u_n - v_n|. \tag{3}$$

Escalator numbers are characterized in the following theorem, which is proved in [4].

Theorem 1. A rational number A is an escalator number if and only if $A^2 - 4A$ is a square in $\mathbb{Q}$.

2. Power Escalator Numbers. In this section, we consider the existence of nontrivial integral powers in the sequence of nonzero escalator numbers $A_2, A_3, \ldots$ determined by a base $a_1 \neq 0, 1$. We start with the squares. Note that by choosing $a_1 \neq 0, 1$ such that $a_1 - 1$ is a perfect square, then $A_2 = a_1^2/(a_1 - 1)$ is a square escalator number. We ask whether there exist examples of square escalator numbers A_n with $n \geq 3$. Our first result shows that the answer is no.

Theorem 2. If a nonzero escalator number A_n is a square, then $n = 2$.

Proof. Suppose that A_n is a square with $n \geq 3$. By equation (1), $A_{n-1} - 1$ is then also a square. Further, since $n \geq 3$, A_{n-1} is also an escalator number and therefore, by Theorem 1, $A_{n-1}^2 - 4A_{n-1}$ is a square. Combining these, $(A_{n-1} - 1)(A_{n-1}^2 - 4A_{n-1})$ is the square of a rational number, say y. Letting $x = A_{n-1} - 2$, we get that $y^2 = (x+1)(x+2)(x-2) = x^3 + x^2 - 4x - 4$. This is an elliptic curve of conductor 48, with $[a_1, a_2, a_3, a_4, a_6] = [0, 1, 0, -4, -4]$ in the Weierstrass form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Using Cremona's tables [1], the group of rational points on this curve is finite of order 4. Thus, the rational points on this curve are $(x, y) = (-2, 0), (-1, 0), (2, 0)$ and the point at infinity. Hence, $A_{n-1} \in \{0, 1, 4\}$. Since A_n is a nonzero escalator number, A_{n-1} is neither 0 nor 1. But if $A_{n-1} = 4$, then $A_{n-1} - 1 = 3$ which is not a square. Hence, if A_n is a square escalator number, then $n = 2$.

We next look at higher powers. We first consider whether a nonzero escalator number can be a fourth power.

Theorem 3. Let $x \in \mathbb{Q} - \{0\}$, then x^4 is not an escalator number.

Proof. Suppose x^4 is a nonzero escalator number. Using equation (2), there exist distinct relatively prime integers, u, v with $v > 0$ such that

$$x^4 = \frac{u^2}{v(u - v)}$$

with $\gcd(u^2, v(u-v)) = 1$. Thus, there are positive integers a, b, and c such that $u = a^2$, $v = b^4$, and $u - v = c^4$. But this yields a nontrivial integral solution to $a^2 = b^4 + c^4$, which is known not to exist (for example, see [5]).

Next, we consider odd powers.

Theorem 4. Let m be an odd integer and $x \in \mathbb{Q} - \{0\}$, then x^m is not an escalator number.

Proof. Suppose that $A_n = x^m$ with $n \geq 2$ and write $A_{n-1} = u/v$ with u and v relatively prime. Then since $x^m = A_n = \frac{u^2}{v(u-v)}$, with $\gcd(u^2, v(u-v)) = 1$, both u^2 and $v(u-v)$ are mth powers. Since m is odd, we get that $u = x_1^m$ for some positive integer x_1. Further, since v and $u-v$ are relatively prime, we get that $v = x_2^m$, $u-v = x_3^m$ for some integers x_2 and x_3. Hence, $x_1^m = u = v + (u-v) = x_2^m + x_3^m$, which, by Fermat's Last Theorem, has no nontrivial integral solution.

The following corollary is immediate.

Corollary 5. Let $m > 2$ and $x \in \mathbb{Q} - \{0\}$, then x^m is not an escalator number.

Thus, the only nontrivial integral power escalator numbers are squares. How common are square escalator numbers? Our next result characterizes the square escalator number and we end the section with a proof that square escalator numbers are dense in the interval $[4, \infty)$.

We will need the following theorem from [4].

Theorem 6. The set of escalator numbers is dense in the set

$$(-\infty, 0] \cup [4, \infty).$$

Theorem 7. The assignment $x \mapsto (x-2)^2$ defines a two-to-one correspondence between the set of escalator numbers and the set of nonzero square escalator numbers.

Proof. Let A be an escalator number. Then, by Theorem 1, $A^2 - 4A$ is a square. Let $y = (A-2)^2$. Then $y^2 - 4y = (A-2)^2(A^2 - 4A)$, which is a square. Hence, again by Theorem 1, $y = (A-2)^2$ is an escalator number.

To see that the assignment is surjective, let $B = z^2 \neq 0$ be a square escalator number, with $z \in \mathbb{Q}$. Then $B(B-4) = B^2 - 4B$ is a square and thus, so is $B - 4$. Then $(z+2)^2 - 4(z+2) = z^2 - 4 = B - 4$ is a square and thus, $z + 2$ is an escalator number.

Note that Theorem 7 shows that all nonzero square escalator numbers are greater than or equal to 4. Hence, the following theorem is optimal, in that the square escalator numbers are not dense in any larger interval.

Theorem 8. The set of nonzero square escalator numbers is dense in the set $[4, \infty)$.

Proof. By Theorem 6, the set E of escalator numbers is dense in $(-\infty, 0] \cup [4, \infty)$. Thus, $E-2$ is dense in $(-\infty, -2] \cup [2, \infty)$ and so the set $(E-2)^2$ is dense in $[4, \infty)$. Theorem 7 completes the proof.

3. Prime Factors. Finally, we look at the prime factors of escalator numbers. Again, for $n \in \mathbb{Z}^+$, let $A_n = u_n/v_n$ with $u_n \in \mathbb{Z}$, $v_n \in \mathbb{Z}^+$, and $\gcd(u_n, v_n) = 1$. For an integer m let $\omega(m)$ be the number of distinct prime factors of m and let $P(m)$ be the largest prime factor of m with the convention that $P(\pm 1) = P(0) = 1$. This section concerns establishing a lower bound for $\omega(u_n v_n)$ and thus for $P(u_n v_n)$.

We begin with a lemma.

Lemma 9. Let $A_n = u_n/v_n$ be a nonzero escalator number, as above. Then

$$\omega(u_n v_n) < \omega(u_{n+1} v_{n+1}).$$

Proof. By equation (3), $u_n | u_{n+1}$ and $v_n | v_{n+1}$. This proves that $\omega(u_n v_n) \leq \omega(u_{n+1} v_{n+1})$.

Suppose $\omega(u_n v_n) = \omega(u_{n+1} v_{n+1})$. Then $\omega(u_n v_n) = \omega(u_n^2 v_n (u_n - v_n)) = \omega(u_n v_n (u_n - v_n))$. Since u_n, v_n, and $u_n - v_n$ are pairwise relatively prime, this implies that $u_n - v_n = \pm 1$. We consider two cases, recalling that $A_{n-1} \neq 0$ or 1. If $u_{n-1} > 0$, then

$$|u_n - v_n| = |u_{n-1}^2 - v_{n-1}(u_{n-1} - v_{n-1})| = |(u_{n-1} - v_{n-1})^2 + u_{n-1} v_{n-1}| \geq 2.$$

If $u_{n-1} < 0$, then

$$|u_n - v_n| = |u_{n-1}^2 - v_{n-1}(u_{n-1} - v_{n-1})| = |u_{n-1}^2 - u_{n-1} v_{n-1} + v_{n-1}^2| \geq 3.$$

Hence, $\omega(u_n v_n) < \omega(u_{n+1} v_{n+1})$.

Theorem 10. Let $A_n = u_n/v_n$ be a nonzero escalator number, as above. Then $\omega(u_n v_n) \geq n-1$ and so $P(u_n v_n) \geq p_{n-1}$, the $n-1$st prime. Further, for $n \geq 11$, $P(u_n v_n) > n \log n$.

Proof. By Lemma 9 and induction, for the first statement, it suffices to prove that for each escalator number A_2, $\omega(u_2 v_2) \geq 2 - 1 = 1$. But by equation (3), $\omega(u_2 v_2) = \omega(u_1^2 v_1 (u_1 - v_1)) \geq 1$, since one of u_1, v_1, and $u_1 - v_1$ must be even.

The final statement follows from the bound $p_k > k \log k + k \log\log k - k$, which holds for $k \geq 2$ [2].

Finally, we note that for all n, $P(u_n) = P(u_1)$, so the above also implies that for sufficiently large n, $P(v_n) > n \log n$.

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