A PROBLEM WITH THE INTERSECTION OF PERIODIC SETS

Russell May

Abstract. We investigate a real-world problem about the measure of the intersection of periodic sets. The problem is modeled in discrete and continuous cases, and a solution is obtained from well-known theorems in undergraduate mathematics.

1. Introduction. When I began graduate school, I moved from the North to the blistering summer heat of Texas. The apartment I rented was cooled by two noisy air conditioners, both of which seemed to be powered by discarded jet engines. After a few days of listening to blasts from one air conditioner or the other (or worse yet *both*), I noticed that each air conditioner seemed to work on a fixed, repetitive cycle—on for a couple minutes, off for a few minutes, on for a couple minutes, and so on. During the rare moments of peace and quiet, I began to wonder about the following problem: Suppose there are two machines, that each machine can be either on or off, and that each machine runs in a periodic fashion. In terms of the fractions of time that each individual machine is off during a single period, what fraction of time are *both* machines off during the tandem period? We consider two cases of this problem, a discrete case and a continuous case. Interestingly enough, solutions to the two cases of this problem involve two great theorems, the Chinese Remainder Theorem [1] from algebra and Birkhoff's Ergodic Theorem [2] from analysis.

2. Discrete Case. In the discrete case, we imagine that each air conditioner AC_i (where $i \in \{1, 2\}$) runs on a cycle of length t_i, where t_i is a positive integer, and that during a cycle, each air conditioner is off during the times enumerated in the set $A_i \subseteq \{1, \ldots, t_i\}$. In this notation, AC_i is off at time t exactly when $(t \bmod t_i) \in A_i$. As an example, AC_1 might have a period of $t_1 = 5$ minutes and be off the first, third, and fifth minutes of each cycle, i.e. $A_1 = \{1, 3, 5\}$. Likewise, AC_2 might have a period of $t_2 = 6$ minutes and be off the first four minutes of the cycle, i.e. $A_2 = \{1, 2, 3, 4\}$. Since the periods t_1 and t_2 are relatively prime, the air conditioners in tandem work on a cycle of $t_1 \cdot t_2 = 30$ minutes, as pictured in Figure 1.

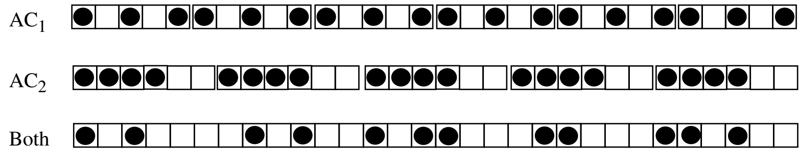

Figure 1. Comparison of individual air conditioners with result in tandem

As the figure indicates, the fraction of time both air conditioners are off is $\frac{12}{30}$, which happens to coincide with the product of the fractions of time when each individual air conditioner is off. To show that this is always the case, recall that for both air conditioners to be off at time t, it must be the case that $(t \mod t_i) \in A_i$ for both $i = 1, 2$. If t_1 and t_2 are relatively prime, the Chinese Remainder Theorem guarantees that for each pair (a_1, a_2) in $A_1 \times A_2$, there is exactly one time t in $\{1, \ldots, t_1 t_2\}$ such that $(t \bmod t_i) \in A_i$ for both air conditioners. This gives a one-to-one correspondence between the quiet times during the tandem period $t_1 t_2$ and $A_1 \times A_2$. So, the fraction of quiet time when neither air conditioner is blasting away is indeed

$$\frac{|A_1|}{t_1} \cdot \frac{|A_2|}{t_2}.$$

3. Continuous Case. In the continuous case, we make the more reasonable assumption that each AC_i runs on a cycle of length t_i, where t_i is a positive real number, and that during the interval $[0, t_i]$ each AC_i is off for the times in a set $A_i \subseteq [0, t_i]$, which we assume to be Lebesgue measurable. For instance, if A_i is the interval (a, b), then the Lebesgue measure of A_i, denoted $m(A_i)$, is just $b-a$. Since each of the air conditioners works periodically, we extend $A_i \subseteq [0, t_i]$ to $A'_i \subseteq \mathbb{R}$ periodically. We say $t \in A'_i$ exactly when $(t \bmod t_i) \in A_i$, where "$t \bmod t_i$" means the number $s \in [0, t_i)$ so that $s - t$ is an integer multiple of t_i. If the periods t_1 and t_2 are incommensurate, i.e. their ratio is irrational, then the on-off cycles of the air conditioners in tandem are not periodic. Therefore, we have to consider the fraction of time when both air conditioners are off as a limit, namely $\lim_{t\to\infty} m(A'_1 \cap A'_2 \cap [0, t])/t$. We can evaluate this limit with the help of Birkhoff's Ergodic Theorem, which says that if $f\colon [0, \tau] \to \mathbb{R}$ is a Lebesgue integrable function and σ is incommensurate with τ, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x + k\sigma \mod \tau) = \frac{1}{\tau} \int_0^\tau f(x')\, dm(x')$, for almost every x. Then, the quantity whose limit is the fraction of time both machines are off is

$$\frac{m(A'_1 \cap A'_2 \cap [0, nt_1])}{nt_1} = \frac{1}{nt_1} \int_0^{nt_1} 1_{A'_1}(t)\, 1_{A'_2}(t)\, dm(t).$$

We can split the integral over the interval $[0, nt_1]$ into a sum of n integrals over intervals of length t_1, and then swap the order of summation and integration, giving

$$\frac{m(A'_1 \cap A'_2 \cap [0, nt_1])}{nt_1} = \frac{1}{t_1}\int_0^{t_1} 1_{A_1}(t)\left(\frac{1}{n}\sum_{k=0}^{n-1} 1_{A_2}(t + kt_1 \mod t_2)\right) dm(t).$$

To obtain the fraction of time when both air conditioners are off, we take the limit as $n \to \infty$ and use Birkhoff's Ergodic theorem to evaluate the sum inside the integral

$$\frac{m(A'_1 \cap A'_2 \cap [0, nt_1])}{nt_1} \to \frac{1}{t_1}\int_0^{t_1} 1_{A_1}(t)\left(\frac{1}{t_2}\int_0^{t_2} 1_{A_2}(s)\, dm(s)\right) dm(t)$$

$$= \frac{m(A_1)}{t_1} \cdot \frac{m(A_2)}{t_2}.$$

So, just as in the discrete case, the fraction of the time when both air conditioners are off is the product of the fractions of the times when each individual machine is off.

4. Generalizations. Despite the fact that a room with more than two periodic air conditioners might be very cold, one can nevertheless wonder if the fraction of quiet time in such a crowded room is still the product of the fractions of the time each machine is off. It turns out that under similar conditions, such as the periods being pair-wise relatively prime or incommensurate, this is true in both the discrete and continuous cases (prove it!). Additionally, in the two-machine case or beyond, the reader is challenged to find a necessary and sufficient condition for the fraction of the time when all air conditioners are off to be the product of the fractions of the times when each individual unit is off.

References

1. I. N. Herstein, *Abstract Algebra*, Wiley, New York, 1999.
2. P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1982.

Mathematics Subject Classification (2000): 11A99, 37A30

Russell May
Department of Mathematics and Computer Science
Morehead State University
150 University Blvd., UPO 701
Morehead, KY 40351-1689
email: r.may@moreheadstate.edu