COMPARISON RESULTS FOR GENERALIZED KOLMOGOROV SYSTEMS WITH RESPECT TO MULTIPLICATIVE SEMIGROUPS

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Abstract. The group $G(\mathcal{A})$ of the invertible elements of a Banach algebra $\mathcal{A}$ can be pre-ordered by a closed semigroup S. We prove a comparison theorem for ODEs of the form $u' = f_1(t,u)u + uf_2(t,u)$ in Banach algebras under the assumption that S is permutation stable. Applications to monotonicity properties of initial value problems and dynamical systems are given.

1. Introduction. Let $(\mathcal{A}, \|\cdot\|)$ be a real Banach algebra with unit $\mathbf{1}$, and let $G(\mathcal{A})$ denote the open group of all invertible elements in $\mathcal{A}$.
Let $S \subseteq \mathcal{A}$ be a closed semigroup, that is $\overline{S} = S$, $\mathbf{1} \in S$, and $x, y \in S \Rightarrow xy \in S$.
A closed semigroup S will be called permutation stable if it has the following property for each $n \in \mathbb{N}$: If $x_1, \ldots, x_n \in G(\mathcal{A})$ then

$$x_{\pi(1)} \cdot \cdots \cdot x_{\pi(n)} \in S$$

either for each permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ or for none. Note that this property holds for each $n \in \mathbb{N}$ if it holds for $n = 3$, and that each closed semigroup is permutation stable in case $\mathcal{A}$ is commutative.
Each closed semigroup $S \subseteq \mathcal{A}$ defines a pre-ordering on $G(\mathcal{A})$: For $x, y \in G(\mathcal{A})$ let

$$x \preceq y \text{ if and only if } x^{-1}y \in S. \qquad (1)$$

For general closed semigroups it is possible to obtain invariance results for equations of the type $u'(t) = f(t, u(t))u(t)$ or $u'(t) = u(t)f(t, u(t))$, see for example L. Markus [3] for invariance of matrix Lie groups.
The aim of this paper is to prove comparison results for ODEs of the form

$$u'(t) = f_1(t, u(t))u(t) + u(t)f_2(t, u(t))$$

in $G(\mathcal{A})$, with respect to pre-orderings induced by permutation stable closed semigroups. Equations of this type can be considered as generalized Kolmogorov systems, since the classical Kolmogorov systems [5] are obtained by concentration on the case $\mathcal{A} = \mathbb{R}^n$ (endowed with the coordinatewise multiplication).

As an introductory example which will illustrate our concepts, consider $\mathcal{A} = M^{n\times n}$, the Banach algebra of all real $n \times n$ matrices. Here

$$S = \{X : 0 \le \det(X) \le 1\}$$

is a permutation stable closed semigroup. Thus, if $X, Y \in M^{n\times n}$ are invertible then $X \preceq Y$ means $0 \le \det(X^{-1}Y) \le 1$.

2. Preliminaries and Notations. In the sequel always let $S \subseteq \mathcal{A}$ be a permutation stable closed semigroup.
Obviously $\preceq$ defined by (1) is a reflexive and transitive relation on $G(\mathcal{A})$, and

$$S \cap G(\mathcal{A}) = \{x \in \mathcal{A} : \mathbf{1} \preceq x\}.$$

Since S is permutation stable $x \preceq y$ if and only if $yx^{-1} \in S$. In particular, if $x \preceq y$ then $y^{-1} \preceq x^{-1}$, since $(y^{-1})^{-1}x^{-1} = yx^{-1} \in S$.
We define

$$W(S) = \{a \in \mathcal{A} : \ \exp(ta) \in S \ (t \ge 0)\}.$$

The set $W(S)$ is a closed wedge, that is $W(S) \neq \emptyset$ (since $\mathbf{1} \in S \Rightarrow 0 \in W(S)$), $\overline{W(S)} = W(S)$ (since S is closed), $\lambda W(S) \subseteq W(S)$ for each $\lambda \ge 0$, and $W(S) + W(S) \subseteq W(S)$ (according to Trotter's product formula).
We consider a second pre-ordering on $\mathcal{A}$. We define

$$a \trianglelefteq b \text{ if and only if } b - a \in W(S).$$

Since $W(S)$ is a wedge, $\trianglelefteq$ is a reflexive and transitive relation on $\mathcal{A}$.
In our introductory example $\mathcal{A} = M^{n\times n}$, $S = \{X : 0 \le \det(X) \le 1\}$, and

$$W(S) = \{A : \mathrm{trace}(A) \le 0\}.$$

According to Wronski's Theorem:

$$\det(\exp(tA)) = \exp(t \cdot \mathrm{trace}\ (A)) \quad (t \ge 0).$$

Thus, $A \trianglelefteq B$ means $\mathrm{trace}(B - A) \le 0$ in this case.

Proposition 1. Let $a, b, x \in \mathcal{A}$. If $\mathbf{1} \preceq x$ and $0 \trianglelefteq a + b$, then

$$\mathbf{1} \preceq \exp(ta)x\exp(tb) \quad (t \ge 0).$$

Proof. Fix $t \ge 0$. We set

$$y_n = \left(\exp\left(\frac{ta}{n}\right)\exp\left(\frac{tb}{n}\right)\right)^n \quad (n \in \mathbb{N}).$$

Each y_n is invertible, and $y_n \to \exp(t(a+b))$ $(n \to \infty)$ according to Trotter's product formula. Hence, $y_n^{-1} \to \exp(-t(a+b))$ $(n \to \infty)$. We have

$$\begin{aligned} \mathbf{1} &\preceq x \exp(t(a+b)) \\ &= x \left(\exp\left(\frac{ta}{n}\right) \exp\left(\frac{tb}{n}\right) \right)^n y_n^{-1} \exp(t(a+b)). \end{aligned}$$

Since S is permutation stable

$$\mathbf{1} \preceq \exp(ta) x \exp(tb) y_n^{-1} \exp(t(a+b)).$$

For $n \to \infty$ we obtain

$$\begin{aligned} \mathbf{1} &\preceq \exp(ta) x \exp(tb) \exp(-t(a+b)) \exp(t(a+b)) \\ &= \exp(ta) x \exp(tb), \end{aligned}$$

since S is closed.

Next, we prove that $W(S)$ is similarity stable.

Proposition 2. If $a, b \in \mathcal{A}$, $0 \trianglelefteq a+b$ and $x \in G(\mathcal{A})$, then $0 \trianglelefteq x^{-1}ax + b$.

Proof. Let $t \geq 0$. We have

$$\mathbf{1} \preceq \exp(t(a+b)) x^{-1} x$$

hence,

$$\mathbf{1} \preceq x^{-1} \exp(t(a+b)) x.$$

Applying Trotter's product formula twice, as in the proof of Proposition 1, leads to

$$\mathbf{1} \preceq x^{-1} \exp(ta) x \exp(tb) = \exp(tx^{-1}ax) \exp(tb),$$

and then to

$$\mathbf{1} \preceq \exp(t(x^{-1}ax + b)).$$

Hence, $0 \trianglelefteq x^{-1}ax + b$.

3. Linear Equations. Let $a, b, x \in \mathcal{A}$, and consider the initial value problem

$$u'(t) = au(t) + u(t)b, \quad u(0) = x. \tag{2}$$

The solution of (2) is

$$u(t) = \exp(ta)x\exp(tb).$$

Proposition 1 says that $u(t) \in S \cap G(\mathcal{A})$ $(t \geq 0)$ if $x \in S \cap G(\mathcal{A})$ and $a + b \in W(S)$. This leads to the invariance condition which will be used later.

Proposition 3. Let $a, b, x \in \mathcal{A}$. If $0 \trianglelefteq a + b$ and $\mathbf{1} \preceq x$, then

$$\lim_{h\to 0+} \frac{1}{h}\mathrm{dist}(x + h(ax + xb), S) = 0.$$

Proof. We have

$$\lim_{h\to 0} \frac{x + h(ax + xb) - u(h)}{h} = 0.$$

By Proposition 1, $u(h) \in S$ $(h \geq 0)$. Therefore,

$$\frac{1}{h}\mathrm{dist}(x + h(ax + xb), S) \leq \frac{\|x + h(ax + xb) - u(h)\|}{h} \to 0$$

as $h \to 0+$.

4. An Invariance Theorem. In the sequel always let $T \in (0, \infty]$. Let E be a Banach space, and let $K(x, \rho)$ denote the open ball in E with center $x \in E$ and radius $\rho > 0$. Let $D \subseteq E$. A function $g\colon [0, T) \times D \to E$ is called locally Lipschitz continuous if to each $(t_0, x_0) \in [0, T) \times D$ there exist numbers $L \geq 0$, $\rho > 0$, $\delta > 0$ such that

$$\|g(t, y) - g(t, x)\| \leq L\|y - x\|$$

for

$$t \in [t_0, t_0 + \delta) \cap [0, T),\ \ x, y \in K(x_0, \rho) \cap D.$$

The following invariance theorem is a special case of an invariance theorem of Volkmann [7], see also the closely related invariance and existence results of Martin [3], Chapter 6.

Theorem 1. Let $D \subseteq E$ be open, $C \subseteq E$ be closed, and let $g\colon [0,T) \times D \to E$ be locally Lipschitz continuous such that

$$\lim_{h\to 0+} \frac{1}{h}\operatorname{dist}(x + hg(t,x), C) = 0 \quad (t \in [0,T),\ x \in D \cap \partial C).$$

If $w\colon [0,T_1) \to D$, $T_1 \le T$ satisfies

$$w'(t) = g(t, w(t))\ (t \in [0,T_1)), \quad w(0) \in C,$$

then $w(t) \in C$ $(t \in [0,T_1))$.

Remark. Theorem 1 is a pure invariance theorem, since g is not assumed to be continuous.

5. Comparison Results. By means of Theorem 1 we prove the following comparison theorem.

Theorem 2. Let $f_1, f_2, f_3, f_4\colon [0,T) \times G(\mathcal{A}) \to \mathcal{A}$ be locally Lipschitz continuous. Let $u_0, v_0 \in G(\mathcal{A})$ with $u_0 \preceq v_0$ and let

$$u\colon [0,T_u) \to G(\mathcal{A}),\ v\colon [0,T_v) \to G(\mathcal{A})$$

be solutions of the initial value problems

$$\begin{aligned} u'(t) &= f_1(t,u(t))u(t) + u(t)f_2(t,u(t)), \quad u(0) = u_0, \\ v'(t) &= f_3(t,v(t))v(t) + v(t)f_4(t,v(t)), \quad v(0) = v_0, \end{aligned} \tag{3}$$

respectively. Assume for each $t \in [0, \min\{T_u, T_v\})$ that

$$f_1(t,u(t)) + f_2(t,u(t)) \trianglelefteq f_3(t,y) + f_4(t,y)\ (u(t) \preceq y), \tag{4}$$

or that

$$f_1(t,x) + f_2(t,x) \trianglelefteq f_3(t,v(t)) + f_4(t,v(t))\ (x \preceq v(t)). \tag{5}$$

Then

$$u(t) \preceq v(t) \quad (t \in [0, \min\{T_u, T_v\})).$$

Proof. We assume (4). Set $J = [0, \min\{T_u, T_v\})$ and consider $w\colon J \to G(\mathcal{A})$ defined by $w(t) = u^{-1}(t)v(t)$. We have $w(0) \in S$, and an easy calculation shows that w is a solution of the differential equation

$$w'(t) = g(t, w(t)) = g_1(t, w(t))w(t) + w(t)g_2(t, w(t))$$

with

$$g, g_1, g_2 \colon J \times G(\mathcal{A}) \to \mathcal{A}$$

defined by

$$g_1(t,x) = (u(t))^{-1}\Big(f_3(t,u(t)x) - f_1(t,u(t))\Big)u(t) - f_2(t,u(t)),$$

$$g_2(t,x) = f_4(t,u(t)x), \quad g(t,x) = g_1(t,x)x + xg_2(t,x).$$

Obviously g_1, g_2 and therefore g are locally Lipschitz continuous. Let $t \in J$ and $x \in S \cap G(\mathcal{A})$. Then $u(t) \preceq u(t)x$, hence, by (4)

$$f_1(t,u(t)) + f_2(t,u(t)) \trianglelefteq f_3(t,u(t)x) + f_4(t,u(t)x),$$

that is

$$0 \trianglelefteq (f_3(t,u(t)x) - f_1(t,u(t))) - (f_2(t,u(t)) - f_4(t,u(t)x)).$$

By Proposition 2

$$0 \trianglelefteq (u(t))^{-1}(f_3(t,u(t)x) - f_1(t,u(t)))u(t) - (f_2(t,u(t)) - f_4(t,u(t)x)),$$

that is $0 \trianglelefteq g_1(t,x) + g_2(t,x)$, and by Proposition 3

$$\lim_{h \to 0+} \frac{1}{h}\operatorname{dist}(x + hg(t,x), S) = 0.$$

According to Theorem 1 $w(t) \in S$ $(t \in J)$, that is $u(t) \preceq v(t)$ $(t \in J)$.
If (5) holds we consider

$$g, g_1, g_2 \colon J \times G(\mathcal{A}) \to \mathcal{A}$$

defined by

$$g_1(t,x) = (u(t))^{-1}\Big(f_3(t,v(t)) - f_1(t,v(t)x^{-1})\Big)u(t) - f_2(t,v(t)x^{-1}),$$

$$g_2(t,x) = f_4(t,v(t)), \quad g(t,x) = g_1(t,x)x + xg_2(t,x).$$

Again $w'(t) = g(t, w(t))$ $(t \in J)$, and g_1, g_2, g are locally Lipschitz continuous since $x \mapsto x^{-1}$ is locally Lipschitz continuous on $G(\mathcal{A})$.
Let $t \in J$ and $x \in S \cap G(\mathcal{A})$. Now, $v(t)x^{-1} \preceq v(t)$, hence by (5)

$$f_1(t, v(t)x^{-1}) + f_2(t, v(t)x^{-1}) \trianglelefteq f_3(t, v(t)) + f_4(t, v(t)),$$

that is

$$0 \trianglelefteq (f_3(t, v(t)) - f_1(t, v(t)x^{-1})) - (f_2(t, v(t)x^{-1}) - f_4(t, v(t))),$$

and with the same conclusion as above

$$\lim_{h \to 0+} \frac{1}{h} \operatorname{dist}(x + hg(t, x), S) = 0.$$

Again Theorem 1 proves $u(t) \preceq v(t)$ $(t \in J)$.

6. Monotone Dependence on the Initial Value. We call a function $f \colon G(\mathcal{A}) \to \mathcal{A}$ monotone increasing if

$$x \preceq y \text{ implies } f(x) \trianglelefteq f(y),$$

and $f \colon [0, T) \times G(\mathcal{A}) \to \mathcal{A}$ is called monotone increasing if $x \mapsto f(t, x)$ is monotone increasing for each $t \in [0, T)$.

Theorem 3. Let $f_1, f_2 \colon [0, T) \times G(\mathcal{A}) \to \mathcal{A}$ be continuous, locally Lipschitz continuous, and let $f_1 + f_2$ be monotone increasing. For each $u_0 \in G(\mathcal{A})$ let

$$u(\cdot, u_0) \colon [0, \omega_+(u_0)) \to G(\mathcal{A})$$

denote the nonextendable solution of the initial value problem (3). If $u_0 \preceq v_0$, then

$$u(t, u_0) \preceq u(t, v_0) \quad (t \in [0, \min\{\omega_+(u_0), \omega_+(v_0)\})).$$

Proof. Apply Theorem 2 with $f_3 := f_1$, $f_4 := f_2$, $u(t) := u(t, u_0)$, and $v(t) := u(t, v_0)$.

7. Monotone Solutions of Dynamical Systems. In analogy to the known results on monotone solutions of dynamical systems with quasimonotone increasing right hand side [2, 6] we have the following theorem.

Theorem 4. Let $g_1, g_2 \colon G(\mathcal{A}) \to \mathcal{A}$ be locally Lipschitz continuous (hence, continuous), and let $g_1 + g_2$ be monotone increasing. Let $u_0 \in G(\mathcal{A})$, and let $u : [0, \omega_+) \to G(\mathcal{A})$ be the nonextendable solution of the dynamical system

$$u'(t) = g_1(u(t))u(t) + u(t)g_2(u(t)), \quad u(0) = u_0.$$

If $0 \trianglelefteq [\ \trianglerighteq\ ]\ g_1(u_0) + g_2(u_0)$, then u is monotone increasing [decreasing] with respect to $\preceq$ on $[0, \omega_+)$.

Proof. First, let $0 \trianglelefteq g_1(u_0)+g_2(u_0)$. Let $f_1, f_2, f_3, f_4 \colon [0, \omega_+) \times G(\mathcal{A}) \to \mathcal{A}$ be defined by $f_1(t, x) = 0,\ f_2(t, x) = 0, f_3(t, x) = g_1(x), f_4(t, x) = g_2(x)$. The solution of $z' = f_1(t, z)z + zf_2(t, z) = 0,\ z(0) = u_0$ is $z(t) = u_0$, and

$$0 = f_1(t, u_0) + f_2(t, u_0) \trianglelefteq f_3(t, y) + f_4(t, y) = g_1(y) + g_2(y)$$

for $t \in [0, \omega_+)$, $u_0 \preceq y$, since $g_1 + g_2$ is increasing. According to Theorem 2 we obtain $u_0 \preceq u(t)$ on $[0, \omega_+)$.
Fix $t_0 \in [0, \omega_+(u_0))$. Now, $v(t) := u(t + t_0)$, $t \in [0, \omega_+ - t_0)$ solves

$$v'(t) = g_1(v(t))v(t) + v(t)g_2(v(t)), \quad v(0) = u(t_0),$$

and $0 \trianglelefteq g_1(u_0) + g_2(u_0) \trianglelefteq g_1(u(t_0)) + g_2(u(t_0))$. Thus, the same argument as above proves $u(t_0) \preceq u(t)$ $(t \in [t_0, \omega_+))$.
In case $0 \trianglerighteq g_1(u_0)+g_2(u_0)$ we consider $g_3, g_4 \colon G(\mathcal{A}) \to \mathcal{A}$ defined by $g_3(x) = -g_1(x^{-1})$, $g_4(x) = -g_2(x^{-1})$. Note that g_3+g_4 is monotone increasing too. Now, $w : [0, \omega_+) \to G(\mathcal{A})$ defined by $w(t) = (u(t))^{-1}$ solves

$$w'(t) = w(t)g_3(w(t)) + g_4(w(t))w(t), \quad w(0) = (u_0)^{-1},$$

and $0 \trianglelefteq g_3((u_0)^{-1}) + g_4((u_0)^{-1})$. According to the first case w is monotone increasing, hence, u is monotone decreasing on $[0, \omega_+)$.

8. Remark on Equations Defined on $[\mathbf{0}, \mathbf{T}) \times \mathcal{A}$. Let $a, b \colon [0, T) \to \mathcal{A}$ be continuous functions, and let $x \in \mathcal{A}$. The initial value problem

$$u'(t) = a(t)u(t) + u(t)b(t), \quad u(0) = x$$

is uniquely solvable on $[0, T)$. By standard reasoning $u(t)$ is invertible for each $t \in [0, T)$ if x is invertible, and in this case $v \colon [0, T) \to G(\mathcal{A})$, $v(t) = (u(t))^{-1}$ is the solution of

$$v'(t) = -b(t)v(t) - v(t)a(t), \quad u(0) = x^{-1}.$$

Therefore, if $f_1, f_2 \colon [0, T) \times \mathcal{A} \to \mathcal{A}$ are continuous and locally Lipschitz continuous, and if

$$u(\cdot, u_0) \colon [0, \omega_+(u_0)) \to \mathcal{A}$$

denotes the nonextendable solution of the initial value problem

$$u'(t) = f_1(t, u(t))u(t) + u(t)f_2(t, u(t)), \quad u(0) = u_0,$$

then $u_0 \in G(\mathcal{A})$ implies $u(t) \in G(\mathcal{A})$ $(t \in [0, \omega_+(u_0)))$. In particular, if $u_0 \in G(\mathcal{A})$, then the maximal interval of existence of the solution does not change if the domain of definition of f_1, f_2 is restricted to $[0, T) \times G(\mathcal{A})$.

9. Examples.

1. First, we return to our introductory example $\mathcal{A} = M^{n\times n}$, $S = \{X : 0 \le \det(X) \le 1\}$. We have seen that $W(S) = \{A : \mathrm{trace}(A) \le 0\}$. In particular $X \preceq Y$ implies $|\det(Y)| \le |\det(X)|$.
Let $A_k, B_k, C_k \in M^{n\times n}$, $k = 1, 2$ with $\mathrm{trace}(B_1 + B_2) \ge 0$ and let $f_k\colon G(M^{n\times n}) \to M^{n\times n}$ be defined by

$$f_k(X) = A_kX - XA_k + |\det(X)|B_k + C_k \quad (k = 1, 2).$$

It is easy to check that $f_1 + f_2$ is monotone increasing. According to Theorem 3 and Theorem 4 the solution of the matrix initial value problem

$$U'(t) = f_1(U(t))U(t) + U(t)f_2(U(t)),\ U(0) = U_0$$

depends increasingly on $U_0 \in G(M^{n\times n})$, and if $U_0 \in G(M^{n\times n})$ and $f_1(U_0) + f_2(U_0) \trianglerighteq [\trianglelefteq] 0$, which means

$$\mathrm{trace}(|\det(U_0)|(B_1 + B_2) + C_1 + C_2) \le [\ge] 0,$$

then $t \mapsto U(t)$ is monotone increasing [decreasing], thus, $t \mapsto |\det(U(t))|$ is monotone decreasing [increasing].

2. Let $\mathcal{A} = \mathbb{R}^4$ endowed with coordinatewise multiplication. We consider the permutation stable closed semigroup

$$S = \{x = (x_1, x_2, x_3, x_4) : x_k \ge 0\ (k = 1, 2, 3, 4),\ x_1x_2 \ge 1, x_3x_4 \ge 1\}.$$

Now, $x \preceq y$ implies $|x_1x_2| \le |y_1y_2|$ and $|x_3x_4| \le |y_3y_4|$. We have

$$W(S) = \{a = (a_1, a_2, a_3, a_4) : a_1 + a_2 \ge 0, a_3 + a_4 \ge 0\}.$$

Let $f\colon G(\mathbb{R}^4) \to \mathbb{R}^4$ be defined by

$$f(x) = \left(-\frac{1}{|x_3x_4|} - x_3^2 - 1, -\frac{1}{|x_1x_2|} + x_3^2 + 1, |x_1x_2x_3x_4| - x_1^3 - 2, x_1^3\right).$$

Again, f is monotone increasing, and $f((1,1,-1,1)) = (-3,1,-2,1) \trianglelefteq 0$. According to Theorem 4 the solution of

$$u'(t) = u(t)f(u(t)), \quad u(0) = (1,1,-1,1)$$

is monotone decreasing. Thus, $t \mapsto |u_1(t)u_2(t)|$ and $t \mapsto |u_3(t)u_4(t)|$ are monotone decreasing.

3. Let $\mathcal{B}$ be any unital normed Banach algebra and let $S_0 = \{x \in \mathcal{B} : \|x\| \leq 1\}$. Then S_0 is a closed semigroup but, in general, not permutation stable. Let $m_+[x,y]$ denote the right hand side derivative of the norm at x in direction y. It is known [1], Chapter 1, that $\|\exp(ta)\| \leq 1$ $(t \geq 0)$ if and only if $m_+[\mathbf{1}, a] \leq 0$.
Let $\mathcal{A}$ be any commutative closed subalgebra of $\mathcal{B}$ containing $\mathbf{1}$, fix $n \in \mathbb{N}$, and let $S = \{x \in \mathcal{A} : \|x^n\| \leq 1\}$. Now S is a permutation stable closed semigroup, and

$$W(S) = \{a \in \mathcal{A} : m_+[\mathbf{1}, a] \leq 0\},$$

since $\|\exp(ta)\| \leq 1$ $(t \geq 0)$ if and only if $\|\exp(nta)\| \leq 1$ $(t \geq 0)$. For example $a - \mu\mathbf{1} \in W(S)$ if $\mu \geq \|a\|$.
Let $c_1, c_2 \colon [0,T) \to \mathcal{A}$ be continuous, $c_1(t) \in W(S)$ $(t \in [0,T))$ and let $f \colon [0,T) \times G(\mathcal{A}) \to \mathcal{A}$ be defined by

$$f(t,x) = \|x^{-n}\| c_1(t) + c_2(t).$$

Then, f is monotone increasing. According to Theorem 3 the solution of

$$u'(t) = (\|(u(t))^{-n}\| c_1(t) + c_2(t)) u(t) \quad u(0) = u_0 \in G(\mathcal{A})$$

depends monotone increasingly on u_0.

4. This time let $\mathcal{B}$ be any unital normed commutative Banach algebra and let $\mathcal{A} = \mathcal{B}^{n \times n}$, the Banach algebra of all $n \times n$-matrices with entries from $\mathcal{B}$. For $A \in \mathcal{A}$ let $\det(A) \in \mathcal{B}$ and $\operatorname{trace}(A) \in \mathcal{B}$ be defined by the common algebraic formulas. It is well-known that $X \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if $\det(X)$ is invertible in $\mathcal{B}$, and that $\det(XY) = \det(X)\det(Y)$ for all $X, Y \in \mathcal{A}$. Since the proof of Wronski's Theorem is purely algebraic too, we also obtain

$$\frac{d}{dt} \det(\exp(tA)) = \operatorname{trace}(A) \det(\exp(tA)) \quad (t \in \mathbb{R}) \tag{6}$$

for each $A \in \mathcal{A}$. Let

$$S := \{X \in \mathcal{A} : \|\det(X)\| \leq 1\}.$$

Since $\mathcal{B}$ is commutative, S is a permutation stable semigroup, and from (6) we obtain

$$W(S) = \{A \in \mathcal{A} : m_+[\mathbf{1}, \mathrm{trace}(A)] \leq 0\}.$$

Moreover $X \preceq Y$ implies $\|\det(Y)\| \leq \|\det(X)\|$, since $\|\det(X^{-1}Y)\| = \|(\det(X))^{-1}\det(Y)\|$.

Now, the functions and conclusions from Example 1 can be reformulated in this setting.

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