# A MARKOV CHAIN FIBONACCI MODEL

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**Abstract.** We present a class of "nice" $n \times n$ Markov probability transition matrices and infinitesimal generators whose limiting (steady state) probabilities are proportional to the first $n$ Fibonacci numbers. We extend this model to other sequences and discover some curious matrix and sequence relationships.

**1. Introduction.** We begin with a particular example of a probability transition matrix $P$ for a Markov chain $\{X_k\}$ with finitely many states $\{0, 1, 2, 3, 4, 5, 6, 7\}$. [1]. Let

$$P = \begin{bmatrix} 2/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix}, \qquad (1)$$

where $p_{ij} = P(X_k = j | X_{k-1} = i)$ for all states $i, j$ and for $k = 1, 2, \ldots$. This represents a fairly natural Markov chain. It looks like a finite one dimensional random walk with equal probabilities of one step to the right or one step to the left except near the endpoints. There is also a probability of a movement directly to zero. So this Markov Chain could represent a population with limited capacity which increases with a birth, decreases with a death, and allows a mass migration of everyone out of the present location, which would lower the population to zero. In studying models, we begin with simple cases so we take all three probabilities to be equal. We stop with a maximum of 7 individuals just to illustrate the results.

The Fibonacci connection to the above matrix, discussed in Section 2, was discovered by chance. We find the limiting probability (steady state) vector for this Markov chain and show that it has Fibonacci type entries. We explain why this happens and find a whole class of transition matrices with similar properties. In Section 3, we extend the state space to the entire set of positive integers. We find the limiting probability vector in two different ways leading to further results. We also consider other types of sequences. In Section 4, we present a few additional comments.

The usual problem in Markov chains is to find the limiting probability vector for a given transition matrix. In some sense, we are solving a reverse problem here of finding a transition matrix which will give a particular limiting vector. The Metropolis-Hastings algorithm and the Gibbs sampler [2] have the same goal but with different purpose and a different type of result.

**2. Limiting Probabilities for a Finite State Space.** To find the limiting vector corresponding to the matrix (1), we define the row vector $\underline{\pi} = (\pi_0, \pi_1, \pi_2, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7)$. Then we solve the system $\underline{\pi} = \underline{\pi}P$ with the additional normalizing condition that $\sum_{i=0}^{7} \pi_i = 1$. We get the results $\pi_6 = 2\pi_7$, $\pi_5 = 5\pi_7$, $\pi_4 = 13\pi_7$, $\pi_3 = 34\pi_7$, $\pi_2 = 89\pi_7$, $\pi_1 = 233\pi_7$, and $\pi_0 = 610\pi_7$.

The first few Fibonacci numbers are

$$
\begin{array}{cccccc}
F_1 = 1 & F_2 = 1 & F_3 = 2 & F_4 = 3 & F_5 = 5 & F_6 = 8 \\
F_7 = 13 & F_8 = 21 & F_9 = 34 & F_{10} = 55 & F_{11} = 89 & F_{12} = 144 \\
F_{13} = 233 & F_{14} = 377 & F_{15} = 610 & F_{16} = 987 & F_{17} = 1597 & F_{18} = 2584.
\end{array}
$$

From $1 = \pi_0 + \cdots + \pi_7$ we obtain $\pi_7 = 1/987$. Thus, the limiting vector is

$$
\begin{aligned}
\underline{\pi} &= \left(\frac{610}{987}, \frac{233}{987}, \frac{89}{987}, \frac{34}{987}, \frac{13}{987}, \frac{5}{987}, \frac{2}{987}, \frac{1}{987}\right) \\
&= \left(\frac{F_{15}}{F_{16}}, \frac{F_{13}}{F_{16}}, \frac{F_{11}}{F_{16}}, \frac{F_9}{F_{16}}, \frac{F_7}{F_{16}}, \frac{F_5}{F_{16}}, \frac{F_3}{F_{16}}, \frac{F_1}{F_{16}}\right).
\end{aligned}
$$

We are surprised to obtain every odd indexed Fibonacci number from $F_{15}$ to $F_1$, each divided by the sum of those Fibonacci numbers, as the limiting probability of a very natural Markov process.

We can find a whole class of probability transition matrices with the same limiting probability vector. We use uniformization methods which convert probability transition matrices for discrete time Markov chains into infinitesimal generators (rate matrices) for continuous time Markov processes and vice-versa. Although some authors (e.g. Medhi [3]) discuss the conversion in both directions, we were unable to find a reference to the result given in Theorem 2.1. (A geometric interpretation for the $2 \times 2$ case appears in Brill and Hlynka [4].)

<u>Theorem 2.1</u>. Let $P$ be an $n \times n$ probability transition matrix for a discrete time Markov chain (DTMC) with limiting vector $\underline{\pi}$. Then the class of matrices of the form $(P - I)/q + I$ has the same limiting vector, where $q$ is a number such that $q \geq max_{i,j}\{q_{ij}\}$, $P - I = [q_{ij}]$, and $I$ is the $n \times n$ identity matrix.

<u>Proof</u>. Since $\underline{\pi} = \underline{\pi}P$, it follows that $\underline{0} = \underline{\pi}(P - I)$. Also, $P - I$ satisfies the conditions of a rate matrix of a continuous time Markov process (CTMP), namely that the rows sum to 0, the off diagonal entries are non-negative, and the diagonal entries are negative. We can divide the entries

of $P-I$ by a real number $q>0$ and still have a rate matrix with the same limiting probability vector. However, we want to convert our rate matrix back to a transition matrix so we need $q$ at least as large as the largest absolute entry in $P-I$. Select any such $q$. Then $\underline{0}=\underline{\pi}(P-I)/q$. Add $\underline{\pi}=I\underline{\pi}$ to both sides to get $\underline{\pi}=\underline{\pi}((P-I)/q+I)$. Because of the way we chose $q$, we know $(P-I)/q+I$ satisfies the conditions of a (DTMC) probability transition matrix, namely that the rows sum to 1 and all the entries lie between 0 and 1. We still have the same limiting vector.

Using Theorem 2.1, we derived a class of probability transition matrices of Fibonacci type. We now present the general result.

Theorem 2.2. Assume $0<b\le 1/3$. Let

$$P=\begin{bmatrix} 1-b & b & 0 & 0 & \dots & 0 & 0 \\ 2b & 1-3b & b & 0 & \ddots & 0 & 0 \\ b & b & 1-3b & b & \ddots & 0 & 0 \\ b & 0 & b & 1-3b & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b & 0 & 0 & \ddots & \ddots & 1-3b & b \\ b & 0 & 0 & 0 & \dots & b & 1-2b \end{bmatrix} \tag{2}$$

be an $n\times n$ transition matrix for a discrete time Markov chain. Then the limiting probability vector is $\underline{\pi}=\left(\frac{F_{2n-1}}{F_{2n}},\frac{F_{2n-3}}{F_{2n}},\dots,\frac{F_1}{F_{2n}}\right)$.

Proof. The states of the Markov chain are labeled $\{0,1,2,\dots,n-1\}$. The balance equation for state $n-1$ comes from the last column. Thus, $\pi_{n-1}=b\pi_{n-2}+(1-2b)\pi_{n-1}$ so $\pi_{n-2}=2\pi_{n-1}$. The balance equations for states $i=1,\dots,n-2$ from columns $2,\dots,n-1$ are of type

$$\pi_i=b\pi_{i-1}+(1-3b)\pi_i+b\pi_{i+1}.$$

This reduces to

$$\pi_{i+1}=3\pi_i-\pi_{i-1}. \tag{3}$$

The theorem claims that the limiting vector is proportional to every other Fibonacci number ordered from largest to smallest. To confirm that our matrix will give Fibonacci type ratios, we need to check the Fibonacci number relationship in reverse of the above order for every second Fibonacci number. Corresponding to (3), we must show $F_{i-2}=3F_i-F_{i+2}$ for all $i$. We note that

$$\begin{aligned} F_{i-2}&=F_i-F_{i-1}=F_i-(F_{i+1}-F_i)\\ &=2F_i-F_{i+1}=2F_i-(F_{i+2}-F_i)=3F_i-F_{i+2}, \end{aligned}$$

as needed. The last column information that $\pi_{n-2} = 2\pi_{n-1}$ corresponds to $F_3 = 2F_1$ for the Fibonacci numbers, and plays the role of the initial condition. Thus, we get the limiting probabilities proportional to the odd indexed Fibonacci numbers. Since $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$ and since the limiting probabilities must sum to 1, i.e., $\pi_0 + \cdots + \pi_{n-1} = 1$, the limiting probability vector follows.

Note that the first column of the transition matrix in the above result does not enter into the calculation of the limiting probability vector. Its information is already accounted for because the rows sum to 1.

In the following corollary, we slightly modify the transition matrix to give a limiting probability vector proportional to even indexed Fibonacci numbers.

Corollary 2.3. Assume $0 < b \leq 1/3$. Let

$$P = \begin{bmatrix} 1-b & b & 0 & 0 & \ldots & 0 & 0 \\ 2b & 1-3b & b & 0 & \ddots & 0 & 0 \\ b & b & 1-3b & b & \ddots & 0 & 0 \\ b & 0 & b & 1-3b & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b & 0 & 0 & \ddots & \ddots & 1-3b & b \\ 2b & 0 & 0 & 0 & \ldots & b & 1-3b \end{bmatrix} \tag{4}$$

be an $n \times n$ transition matrix for a discrete time Markov chain. Then the limiting probability vector is $\underline{\pi} = \left(\frac{F_{2n}}{F_{2n+1}-1}, \frac{F_{2n-2}}{F_{2n+1}-1}, \ldots, \frac{F_2}{F_{2n+1}-1}\right)$.

Proof. We note that columns $2, \ldots, n-1$ are the same as in Theorem 2.1. The last column is changed to reflect the change in the initial condition and the first column is changed so that the rows sum to 1. The last column implies that $\pi_{n-2} = 3\pi_{n-1}$ and this corresponds to the initial condition of Fibonacci numbers that $F_2 = 1$ and $F_4 = 3$. The result follows.

Our next objective is to find a transition matrix which will give ALL of the first $n$ Fibonacci numbers, rather than every other one. Determining such a matrix causes some difficulty. It turns out that we cannot find a matrix of the same type used to generate alternating Fibonacci numbers, but there is a solution with a similar type of transition matrix. The new matrix has a subdiagonal with almost all zeros. Our result is as follows.

Theorem 2.4 Assume $0 < b \leq 1/2$. Let

$$P = \begin{bmatrix} 1-b & b & 0 & 0 & \dots & 0 & 0 & 0 \\ b & 1-2b & b & 0 & \ddots & 0 & 0 & 0 \\ b & 0 & 1-2b & b & \ddots & \ddots & 0 & 0 \\ 0 & b & 0 & 1-2b & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 1-2b & b & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 & 1-2b & b \\ 0 & 0 & 0 & 0 & \dots & b & 0 & 1-b \end{bmatrix} \tag{5}$$

be an $n \times n$ transition matrix for a discrete time Markov chain. Then the limiting probability is $\underline{\pi} = \left(\frac{F_n}{F_{n+2}-1}, \frac{F_{n-1}}{F_{n+2}-1}, \dots, \frac{F_1}{F_{n+2}-1}\right)$.

Proof. The states of the Markov chain are labeled $\{0, 1, 2, \dots, n-1\}$. The balance equation for state $n-1$ comes from the last column and represents the initial condition for Fibonacci recursion. Thus, $\pi_{n-1} = b\pi_{n-2} + (1-b)\pi_{n-1}$ so $\pi_{n-2} = \pi_{n-1}$. The balance equation for state $n-2$ comes from the second last column and implies that $\pi_{n-3} = 2\pi_{n-2} = 2\pi_{n-1}$. The balance equations for states $1, \dots, n-3$ come from columns $2, \dots, n-2$ and are of type

$$\pi_i = b\pi_{i-1} + (1-2b)\pi_i + b\pi_{i+2}.$$

This reduces to

$$\pi_{i+2} = 2\pi_i - \pi_{i-1}. \tag{6}$$

Since the $\pi_i$'s are giving the Fibonacci ratios in reverse order, we must show that $F_{i-1} = 2F_{i+1} - F_{i+2}$ for all $i$. If we can show that Fibonacci numbers satisfy this property, then we can conclude that the limiting probabilities must be proportional to Fibonacci numbers. In fact

$$2F_{i+1} - F_{i+2} = 2F_{i+1} - F_{i+1} - F_i = F_{i+1} - F_i = F_{i+1} - F_{i+1} + F_{i-1} = F_{i-1}$$

as required. The initial conditions are implied by the final two columns. But $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ and since the limiting probabilities must sum to 1, the desired limiting probability form follows.

We consider the special case of $b = 1/2$ in the Fibonacci matrix in Theorem 2.4. Then the matrix has 0 entries on most of the diagonal, and the form is quite simple. For example, if $n = 5$, our $n \times n$ matrix becomes

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

The limiting vector in this case is $(5/12, 3/12, 2/12, 1/12, 1/12)$.

In the above example, and in our earlier results, the limiting probability vector has entries in descending order of magnitude. The states are currently named $\{0, 1, 2, \ldots, n-2, n-1\}$. If we simply reverse the names of the states by $i \to n-1-i$, the new transition matrix in the previous example would be

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

and the limiting vector is in ascending order $(1/12, 1/12, 2/12, 3/12, 5/12)$. This new matrix is obtainable by reflecting the original entries over the center of the matrix.

If we consider the Fibonacci transition matrix $P$ given by (5), there is a rate matrix for a continuous time Markov process with the same limiting probability. That matrix is

$$P - I = \begin{bmatrix} -b & b & 0 & 0 & \ldots & 0 & 0 & 0 \\ b & -2b & b & 0 & \ddots & 0 & 0 & 0 \\ b & 0 & -2b & b & \ddots & \ddots & 0 & 0 \\ 0 & b & 0 & -2b & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & -2b & b & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 & -2b & b \\ 0 & 0 & 0 & 0 & \ldots & b & 0 & -b \end{bmatrix}. \tag{7}$$

This looks very close to the rate matrix for an $E_2/M/1$ queueing system, [3] for $\rho = 1$, except for the second row. It also looks like a rate matrix for a queueing system with individual arrivals and bulk service (of size 2). One possible description of the model allows a server to serve a single customer if only one customer is in the system, but must serve two customers at a time (at the same rate as for one customer) if there are at least two customers available for service. So we have the remarkable result that for a very natural queueing system (with a finite buffer), the limiting probabilities have Fibonacci ratios.

In Mandelbaum, Hlynka, and Brill [6], it was observed that any probability distribution has a birth and death representation. Let $F_i$ be the

Fibonacci numbers. Consider a transition diagram of a birth and death process of type:

$$\begin{array}{ccccccccc} & F_{n-1} & & F_{n-2} & & F_{n-3} & \dots & F_1 & \\ & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & \\ 0 & & 1 & & 2 & & \dots & & n-1 \\ & \leftarrow & & \leftarrow & & \leftarrow & & \leftarrow & \\ & F_n & & F_{n-1} & & F_{n-2} & \dots & F_2 & \end{array}$$

This system has a rate matrix

$$\begin{bmatrix} -F_{n-1} & F_{n-1} & 0 & 0 & \dots & 0 \\ F_n & A_n & F_{n-2} & 0 & \ddots & 0 \\ 0 & F_{n-1} & A_{n-1} & F_{n-3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & A_3 & F_1 \\ 0 & 0 & 0 & \dots & F_2 & -F_2 \end{bmatrix}, \tag{8}$$

where $A_i = -F_i - F_{i-2}$.

The limiting vector for this rate matrix is exactly the same as for the rate matrix (7) and for the transition matrix (5). However, there are major differences in the form of the matrices (7) and (8). First (8) is the rate matrix of a birth and death process and is a tridiagonal matrix. Also, the components are already Fibonacci numbers so it is not surprising that the limiting vector yields Fibonacci numbers. By contrast, in the much more interesting rate matrix (7), there are no Fibonacci entries, and the matrix corresponds to a fairly natural queueing system, yet the limiting vector is the same as that of (8). Thus, we can have more than one class of rate matrix (or transition matrix) that generates Fibonacci type limiting vectors, but some matrix classes are "nicer" than others.

We found one further interesting transition matrix which gives the first $n$ Fibonacci numbers in the limiting vector, but in a strange order. We present a specific example when $n = 7$. Let

$$P = \begin{bmatrix} .7 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & .7 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & .7 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 & .3 & 0 & 0 \\ 0 & 0 & .3 & .3 & .1 & .3 & 0 \\ 0 & .3 & 0 & 0 & .3 & .1 & .3 \\ .6 & 0 & 0 & 0 & 0 & .3 & 1 \end{bmatrix}.$$

The limiting probability vector in this case is $(1/53)(2, 5, 13, 21, 8, 3, 1)$. Note the alternating Fibonacci numbers in the vector. This particular class of threshold matrices has some potential applications in medicine.

Since the recursion relationship for the Lucas numbers is the same as that of the Fibonacci numbers, we should be able to get a corresponding probability transition matrix by modifying the last two columns of (5). We find that the following matrix gives the Lucas numbers.

$$\begin{bmatrix} 1-b & b & 0 & 0 & \dots & 0 & 0 & 0 \\ b & 1-2b & b & 0 & \ddots & 0 & 0 & 0 \\ b & 0 & 1-2b & b & \ddots & \ddots & 0 & 0 \\ 0 & b & 0 & 1-2b & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 1-2b & b & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 & 1-3b & 2b \\ 0 & 0 & 0 & 0 & \dots & b & 0 & 1-b \end{bmatrix}. \tag{9}$$

The tribonacci numbers are defined by $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n = 4, \dots$. We notice in (5) that there is a subdiagonal of (almost all) zeros. If we modify the lower triangular part by inserting an extra subdiagonal of zeros, we get yet another Markov transition matrix. This new matrix is

$$\begin{bmatrix} 1-b & b & 0 & 0 & \dots & 0 & 0 & 0 \\ b & 1-2b & b & 0 & \ddots & 0 & 0 & 0 \\ b & 0 & 1-2b & b & \ddots & 0 & 0 & 0 \\ b & 0 & 0 & 1-2b & \ddots & \ddots & 0 & 0 \\ 0 & b & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \dots & 0 & 1-2b & b \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1-b \end{bmatrix} \tag{10}$$

and has limiting vector $\left(\frac{T_n}{S}, \frac{T_{n-1}}{S}, \dots, \frac{T_1}{S}\right)$, where $S = \sum_{i=1}^{n} T_n$. In a similar manner we can define and generate tetranacci numbers, pentanacci numbers, and so on. The corresponding rate matrices $P - I$ model finite buffer queueing systems with bulk service where the bulk size for tribonacci numbers is 3, for the tetranacci numbers is 4, and so on.

**3. The Infinite State Space** We next consider the infinite state space $\{0, 1, 2, \dots\}$ with the same matrix structure as in (2). First note that

$$G = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

is the famous "golden ratio." $G$ satisfies $G^2 - G - 1 = 0$ and $G = (\sqrt{5}+1)/2$. We obtain the following pretty theorem.

Theorem 3.1. Assume $0 < b \leq 1/3$. Let

$$P = \begin{bmatrix} 1-b & b & 0 & 0 & 0 & \dots \\ 2b & 1-3b & b & 0 & 0 & \ddots \\ b & b & 1-3b & b & 0 & \ddots \\ b & 0 & b & 1-3b & b & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{11}$$

be a transition matrix for a discrete time Markov chain. Then the limiting probability vector is $\underline{\pi} = (L, L^3, L^5, \dots)$, where

$$L = \lim_{n\to\infty} \frac{F_n}{F_{n+1}},$$

$L$ satisfies $L^2 + L - 1 = 0$, and $L = (\sqrt{5} - 1)/2$ is the reciprocal of the golden ratio.

Proof. First we note that the Markov chain defined by the transition matrix is positive recurrent for $0 < b < 1/3$. Next, we look at the first coordinate $F_{2n-1}/F_{2n}$ from the result in Theorem 2.2. Thus, the first coordinate in the limiting vector in the infinite state space case is

$$\lim_{n\to\infty} \frac{F_{2n-1}}{F_{2n}} = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = L.$$

The second coordinate is

$$\lim_{n\to\infty} \frac{F_{2n-3}}{F_{2n}} = \lim_{n\to\infty} \frac{F_{2n-3}}{F_{2n-2}} \frac{F_{2n-2}}{F_{2n-1}} \frac{F_{2n-1}}{F_{2n}} = L^3.$$

Continuing in this way, we get our result in terms of $L$. The fact that the sum of the probabilities must equal 1 gives a quadratic in $L$ from which $L$ is determined.

We also attempted to find the limiting vector using conventional techniques. This created a difficulty but also revealed a known result about Fibonacci numbers.

Theorem 3.2. Let $L = (\sqrt{5}-1)/2$ be the reciprocal of the golden ratio. Then $L^{2n+1} = F_{2n+1}L - F_{2n}$ for $n = 1, 2, \dots$.

Proof. This result can be proved directly in a fairly simple manner by using the fact that $L^2 = 1 - L$ to lower the powers. However, we wish to indicate how we rediscovered the result using probability methods. The balance equations from (11) are

$$\pi_0 = (1-b)\pi_0 + 2b\pi_1 + b\pi_2 + b\pi_3 + \cdots \tag{12}$$
$$\pi_1 = b\pi_0 + (1-3b)\pi_1 + b\pi_2 \tag{13}$$
$$\pi_2 = b\pi_1 + (1-3b)\pi_2 + b\pi_3 \tag{14}$$
$$\ldots .$$

From (12) and the fact that $\sum_{i=0}^{\infty} \pi_i = 1$, we obtain $\pi_1 = 2\pi_0 - 1$. From (13), we obtain $\pi_2 = 3\pi_1 - \pi_0 = 5\pi_0 - 3$. An induction argument gives $\pi_n = F_{2n+1}\pi_0 - F_{2n}$. From Theorem 3.1, $\pi_n = L^{2n+1}$ for $n = 0, 1, \ldots$. Equating the two expressions for $\pi_n$ gives the result.

Note.

(1) The limiting case of the transition matrix from the even indexed Fibonacci numbers is exactly the same as for the odd indexed case. The limiting vector in both cases has components which are just powers of $L$.

(2) One standard method of finding the limiting probabilities is to use generating functions. If we define $\phi(z) = \sum_{i=0}^{\infty} \pi_i z^i$, then we can obtain an expression for $\phi(z)$ by multiplying (12) by $z^0$, (13) by $z^1$, (14) by $z^2$, and so on. Summing both sides and solving for $\phi(z)$ yields

$$\phi(z) = \frac{z - \pi_0(1-z)}{z - (1-z)^2}.$$

This still leaves the difficulty of finding $\pi_0$, so it is fortunate that we already used the limit of the finite case to give us $\pi_0 = L$. Kleinrock uses the roots of the denominator of the generating function to obtain the value of $\pi_0$. Our method is equivalent but perhaps more accessible.

In a similar manner to the previous results, we can find the limiting vector for the infinite state transition matrix (corresponding to (5)).

Theorem 3.3. Assume $0 < b \leq 1/2$. Let

$$P = \begin{bmatrix} 1-b & b & 0 & 0 & 0 & \cdots \\ b & 1-2b & b & 0 & 0 & \ddots \\ b & 0 & 1-2b & b & 0 & \ddots \\ 0 & b & 0 & 1-2b & b & \ddots \\ 0 & 0 & b & 0 & 1-2b & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{15}$$

be a transition matrix for a discrete time Markov chain. Then the limiting probability vector is $\underline{\pi} = (L^2, L^3, L^4, \dots)$, where

$$L = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{\sqrt{5}-1}{2}$$

is the reciprocal of the golden ratio.

<u>Proof</u>. The proof follows in the same manner as in Theorem 3.1.

For the Tribonacci number matrix extended to the space of all non-negative integers, we have the following result.

<u>Theorem 3.4</u>. The limiting vector corresponding to the Markov transition matrix

$$\begin{bmatrix} 1-b & b & 0 & 0 & 0 & \cdots \\ b & 1-2b & b & 0 & 0 & \ddots \\ b & 0 & 1-2b & b & 0 & \ddots \\ b & 0 & 0 & 1-2b & b & \ddots \\ 0 & b & 0 & 0 & 1-2b & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{16}$$

is $(1-M, (1-M)M, (1-M)M^2, \dots)$, where $M \ge 0$ satisfies $1 = x + x^2 + x^3$.

<u>Proof</u>. In the finite case, the matrix (10) yields $\left(\frac{T_n}{S}, \frac{T_{n-1}}{S}, \dots, \frac{T_1}{S}\right)$, where $S = \sum_{i=1}^n T_i$. We know

$$\lim_{n\to\infty} \frac{T_n}{S} = \alpha$$

exists. Let

$$M = \lim_{n\to\infty} \frac{T_{n-1}}{T_n}.$$

Now $T_{n+1} = T_n + T_{n-1} + T_{n-2}$. Divide by $T_{n+1}$ to get $1 = M + M^2 + M^3$. Then

$$\lim_{n\to\infty} \frac{T_{n-1}}{S} = \alpha M$$

and so on. Thus, the limiting vector is $(\alpha, \alpha M, \alpha M^2, \dots)$. But the limiting probabilities sum to 1 so $1 = \dfrac{\alpha}{1-M}$. Thus, $\alpha = 1 - M$ and the result follows.

**4. Conclusion and Acknowledgment.** In this paper we have illustrated that the Fibonacci numbers (and variants) make their appearance in the limiting vector of a class of natural Markov transition matrices and infinitesimal generators. We showed that a fixed Fibonacci type limiting vector can arise from more than one type of transition matrix. Our methods allow us to obtain limiting vectors for certain infinite state processes in a relatively easy manner, by working with properties of the finite state version. Traditionally, rate matrices for birth and death processes have been a major focus of probability models and will continue to have that role. Hopefully, our presentation will encourage other models to be examined more carefully. Beyond the work of this paper, we have found other matrices giving various forms of Fibonacci type sequences, but this paper has presented the most interesting relationships that we have discovered. Generalizations of some of the material appear in Sajobi [7].

Unanswered questions include the following. Can we obtain a "nice" transition matrix such that the limiting vector gives every third Fibonacci number, every fourth Fibonacci number, and so on? Given an infinite state space transition matrix, under what circumstances can we finitely truncate (with adjustments) to get a sufficiently "nice" finite state limiting vector, which can be used to obtain the infinite state limiting vector in an easy manner? Are there other nice queueing models that give surprisingly nice limiting probability vectors?

Special cases of the matrices discussed in this paper can be examined easily with computational packages, such as MAPLE and MATLAB. A large power of the matrix will make all the rows equal to the limiting probability vector. Dividing the limiting probability vector by the minimal entry will give a vector of Fibonacci or other types of numbers, as indicated by the theorems.

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