ON GENERALIZED ALTERNATING GALILEO SEQUENCES

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Abstract. During the course of studying freely falling bodies, Galileo observed a wonderful property that is unique to the sequence of positive odd integers. In this paper, we present and explore a generalized variation of this classical result. We exhibit some properties of our variation and leave the reader with some open questions for possible further exploration.

1. Background. For each positive integer n, consider the following identity

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \frac{1+3+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(4n-1)} = \cdots . \tag{1.1}$$

Galileo first observed this property of positive odd integers in relation to his work on falling bodies [2,3]. Identity (1.1) may be interpreted as follows. The distance traveled by a falling body in the first n units of time is always one-third the distance in the next n units of time. From what we can surmise since then, not much has been written about this beautiful identity. Noteworthy exceptions are generalizations to a family of *Galileo sequences* [5] and a combinatorial proof without words [4] of (1.1). The identity is a prime example of the sort of mathematical fact that seems innocuous and perhaps even mildly interesting at first, but then anyone willing to delve into its depths quickly uncovers wonderful balance and harmony. Galileo *observed* that the sequence of positive odd integers was the only integer-valued sequence in arithmetical progression with this property. It is not clear how Galileo proved this and so we present an elementary proof of a generalized version of Galileo's result. We also define a generalized alternating version of (1.1) and present analogous results. In conclusion, we offer some open problems for further exploration.

2. Definitions. For any positive, real-valued sequence a_n, consider the n^{th} partial sum $S_n = a_1 + a_2 + \cdots + a_n$. Then a_n is called a (standard) *Galileo sequence* (GS) if for each positive integer n we have $S_{2n} - S_n = cS_n$ for some fixed (independent of n) constant c. Thus, (1.1) implies that the sequence of positive odd integers is an example of a Galileo sequence. So, what about sequences where not all terms are positive? To answer this question, we propose a modified definition of a Galileo sequence. An (ordinary) *alternating even (respectively, odd) Galileo sequence* (AEGS, respectively AOGS) is one where $S_n = a_1 - a_2 + \cdots + (-1)^{(n-1)}a_n$ and we have $S_{2n} - S_n = c_e S_n$ for all even (respectively, odd) n and for some constant c_e (respectively, c_o). Next, we define a generalized version. For each positive integer k, we call $a_n = a_n(k)$ a *k-alternating even Galileo sequence* (k-AEGS) if the nth partial sum is given by

$$S_n(k) = (a_1 + a_2 + \cdots + a_k) - (a_{k+1} + a_{k+2} + \cdots + a_{2k}) + \cdots \\ + (-1)^{(n-1)}(a_{(n-1)k+1} + a_{(n-1)k+2} + \cdots + a_{nk})$$

and we have $S_{2n}(k) - S_n(k) = c_e S_n(k)$ for all even n and some constant c_e. Analogously, we define a k-AOGS. We will refer to the expression $(a_1 + a_2 + \cdots + a_k)$ as the 1-block of $S_n(k)$, $(a_{k+1} + a_{k+2} + \cdots + a_{2k})$ as the 2-block of $S_n(k)$, etc. A k-alternating Galileo sequence that is both even and odd is said to be simply a k-AGS. Clearly, a 1-AGS corresponds to the ordinary AGS. Note that we do not require that $c_e = c_o$ (for otherwise, the sequence is a Galileo sequence by definition). It is easy to see that if a sequence is a GS or AGS, then any multiple of it is also a GS or AGS, respectively. Hence, it makes sense to define a *primitive* GS (or AGS) as one that is minimal with respect to scaling [3]. For the remainder of this paper, all Galileo sequences will be assumed to be primitive.

3. Results.

3.1. Alternating Galileo Sequences. We begin with a generalization of the classical result due to Galileo.

Theorem 1. Let a_n be a real-valued sequence in arithmetical progression with common difference d. Then a_n is a Galileo sequence if and only if $c = 3$ and $d = 2a_1$.

Proof. First, recall that the nth partial sum of a sequence a_n in arithmetical progression is given by $S_n = \frac{n}{2}[2a_1 + (n-1)d]$. Consequently, $S_{2n} - S_n = \frac{n}{2}[2(a_1 + nd) + (n-1)d]$.

($\Rightarrow$) If a_n is a Galileo sequence, then by definition, there exists a constant c such that

$$\frac{n}{2}[2(a_1 + nd) + (n-1)d] = c\frac{n}{2}[2a_1 + (n-1)d] \\ \Leftrightarrow (2a_1 + 3nd - d) = 2ca_1 + cnd - cd \\ \Leftrightarrow d = \frac{2a_1(c-1)}{n(3-c) + c - 1}.$$

We now note that the common difference d is independent of n if and only if $c = 3$. Finally, it is easy to see from the last equation above that if $c = 3$, then $d = 2a_1$.

($\Leftarrow$) This follows from the definition of a GS.

Corollary 1. (Galileo) The sequence of positive odd integers is the only primitive GS in arithmetical progression.

Proof. Choose $a_1 = 1$ in Theorem 1.

Next, we explore some properties of a generalized alternating version of the sequence of positive odd integers and find results that are analogous to Galileo's original observation.

Lemma 1. For each positive integer k, if $a_n(k) = (-1)^{(n-1)}[\{2((n-1)k+1)-1\} + \{2((n-1)k+2)-1\} + \cdots + \{2nk-1\}]$, then $S_n(k) = (-1)^{n-1}k^2 n$.

Proof. The proof is by induction on n. Note that each block of $S_n(k)$ consists of k consecutive odd integers. Using the definition of $S_n(k)$, for $n = 1$, we have $S_1(k) = a_1 + a_2 + \cdots + a_k = 1 + 3 + \cdots + (2k-1) = k^2$. On the other hand, $(-1)^{(1-1)}k^2(1) = k^2$. This establishes the base case. Next, assume that for arbitrary but fixed n, we have $S_n(k) = (-1)^{n-1}k^2 n$ and consider the $(n+1)^{\text{st}}$ partial sum $S_{(n+1)}(k)$. Again, by the definition of $S_n(k)$, we have

$$\begin{aligned} S_{(n+1)k} &= S_n(k) + a_{(n+1)}k \\ &= (-1)^{n-1}k^2 n + (-1)^n[\{2(nk+1)-1\} + \{2(nk+2)-1\} \\ &\quad + \cdots + \{2(nk+k)-1\}] \\ &= (-1)^{n-1}[k^2 n - (2nk^2 + (1 + 3 + \cdots + (2k-1))] \\ &= (-1)^{n-1}[-nk^2 - k^2] \\ &= (-1)^n k^2(n+1), \end{aligned}$$

as required. This completes the proof.

Lemma 2. For a sequence $a_n(k)$ as defined in Lemma 1, we have

$$S_{2n}(k) - S_n(k) = \begin{cases} -k^2 n, & \text{if } n \text{ is even;} \\ -3k^2 n, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From Lemma 1, $S_n(k) = (-1)^{n-1}k^2 n$. Hence, $S_{2n}(k) - S_n(k) = (-1)^{2n-1}k^2(2n) - (-1)^{n-1}k^2 n$. Now, we see that if n is even, then $S_{2n}(k) - S_n(k) = -2k^2 n + k^2 n = -k^2 n$. On the other hand, if n is odd, then $S_{2n}(k) - S_n(k) = -2k^2 n - k^2 n = -3k^2 n$.

Theorem 2. For each positive integer k, the sequence given by $a_n(k) = (-1)^{(n-1)}[\{2((n-1)k+1)-1\} + \{2((n-1)k+2)-1\} + \cdots + \{2nk-1\}]$, consisting of k-blocks of alternating consecutive odd integers is a k-AGS.

Proof. For any k, if n is even, then from Lemmas 1 and 2, we observe that $S_n(k) = S_{2n}(k) - S_n(k) = -k^2 n$. Hence, the sequence $a_n(k)$ is a k-AEGS with $c_e = 1$. On the other hand, if n is odd, then again from Lemmas 1 and 2, we see that $S_{2n}(k) - S_n(k) = -3S_n(k)$. Thus, the sequence $a_n(k)$ is a k-AOGS with $c_o = -3$. Consequently, by definition, $a_n(k)$ is a k-AGS.

To get a better feeling for Theorem 2, we provide an example.

Example 1. Let $k = 3$. Then Theorem 2 states that for even n, we have

$$\begin{aligned}
&\frac{(1+3+5)-(7+9+11)}{(13+15+17)-(19+21+23)} \\
&= \frac{(1+3+5)-(7+9+11)+(13+15+17)-(19+21+23)}{(25+27+29)-(31+33+35)+(37+39+41)-(43+45+47)} \\
&= \cdots = 1
\end{aligned}$$

and, for odd n, we have

$$\begin{aligned}
&\frac{(1+3+5)}{-(7+9+11)} \\
&= \frac{(1+3+5)-(7+9+11)+(13+15+17)}{-(19+21+23)+(25+27+29)-(31+33+35)} \\
&= \cdots = -\frac{1}{3}.
\end{aligned}$$

Next, we develop another familiar example of an alternating even (but not odd) Galileo sequence. This time consider a generalized sequence where each k-block of a partial sum consists of a sum of k consecutive positive integers. In other words, each block is the sum of the terms of a subsequence of a sequence corresponding to a triangular number [1].

Lemma 3. For each positive integer k and n, if $a_n = (-1)^{n-1}[\{(n-1)k+1\} + \{(n-1)k+2\} + \cdots + \{nk\}]$, then $S_n(k) = -\frac{k^2 n}{2}$ for all even values of n.

Proof. Just like Lemma 1, this result is a standard proof by induction. Of course, since the formula for $S_n(k)$ is valid for only *even* values of n, the base case corresponds to $n = 2$. In this case,

$$\begin{aligned}
S_2(k) &= (1+2+\cdots+k) - ((k+1)+(k+2)+\cdots+(2k)) \\
&= (1+2+\cdots+k) - k(k) - (1+2+\cdots+k) \\
&= -k^2 = (-1)^{2-1}\frac{k^2 \cdot 2}{2}.
\end{aligned}$$

This establishes the base case. Next, assume that for some arbitrary (but fixed) positive integer m, we have

$$S_{2m}(k) = -\frac{k^2 2m}{2} = -k^2 m,$$

and consider the partial sum $S_{2m+2}(k)$. Observe that

$$\begin{aligned}
&S_{2m+2}(k) = S_{2m}(k) + a_{2m+1}(k) + a_{2m+2}(k)\\
&= -k^2m + (-1)^{2m}[(2mk+1) + \cdots + (2mk+k)]\\
&\quad + (-1)^{2m+1}[\{(2m+1)k+1\} + \cdots + \{(2m+1)k+k\}]\\
&= -k^2m + [2mk^2 + (1+2+\cdots+k)] - [(2m+1)k^2 + (1+2+\cdots+k)]\\
&= -k^2m + 2mk^2 - (2m+1)k^2 = -k^2m - k^2\\
&= -k^2(m+1) = -\frac{k^2(2m+2)}{2},
\end{aligned}$$

as required and this completes the proof.

Now, we are ready to exhibit another example of a k-AEGS.

<u>Theorem 3</u>. For each positive integer k, the sequence given by $a_n = (-1)^{n-1}[\{(n-1)k+1\} + \{(n-1)k+2\} + \cdots + \{nk\}]$, consisting of k-blocks of alternating consecutive integers is a k-AEGS with $c_e = 1$.

<u>Proof</u>. From Lemma 3, for each positive integer k and m, we observe that

$$\begin{aligned}
S_{4m}(k) - S_{2m}(k) &= -\frac{k^2(4m+2)}{2} + \frac{k^2(2m+2)}{2}\\
&= -k^2(2m+1) + k^2(m+1)\\
&= -k^2m\\
&= S_{2m}(k),
\end{aligned}$$

which completes the proof.

However, it can be easily checked that the sequence $a_n(k)$ defined in Lemma 3 is *not* a k-AOGS and consequently, not a k-AGS. As an illustration of Theorem 3, consider the following example.

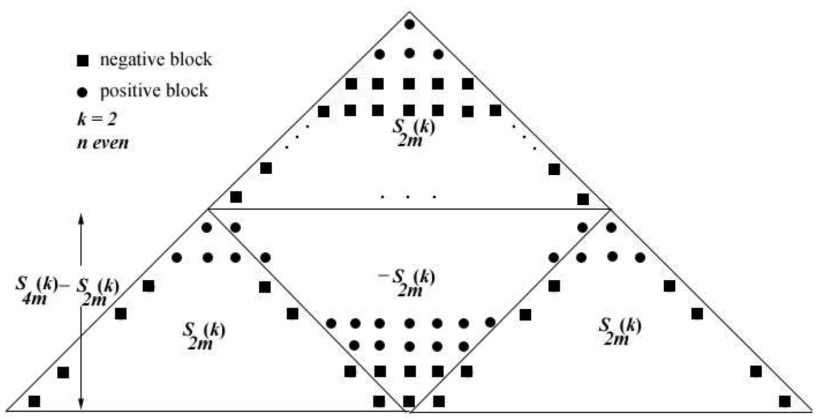
<u>Example 2</u>. Let $k = 4$ and for m and n positive integers let

$$S_{m,n} = \sum_{i=m}^{n} i.$$

Then, Theorem 3 states that when n is even, we have

$$\begin{aligned}
&\frac{S_{1,4} - S_{5,8}}{S_{9,12} - S_{13,16}}\\
&= \frac{S_{1,4} - S_{5,8} + S_{9,12} - S_{13,16}}{S_{17,20} - S_{21,24} + S_{25,28} - S_{29,32}}\\
&= \cdots = 1.
\end{aligned}$$

3.2 Combinatorial Proofs. As is common with many results on properties of integers that have inductive proofs, it turns out that Theorems 3 and 4 may also be proved using a combinatorial argument commonly referred to as a *proof without words*. We give one such proof and leave the others for the reader to verify. In the context of alternating Galileo sequences, Figure 1 *proves* Theorem 2 in the case where $k = 2$ and n is even. To see this, note that the *lower half* of the diagram is a partition of $S_{2n}(k) - S_n(k)$ into three parts that sum to just $S_n(k)$. In other words, $S_{2n}(k) - S_n(k) = S_n(k)$, proving Theorem 2 for even n and $k = 2$. In general, the other proofs without words for all other cases involve *rearranging* the terms of $S_{2n}(k) - S_n(k)$ as desired.



Proof Without Words of Theorem 2 for $k = 2$ and even n.

4. Concluding Remarks. The main purpose of this paper is to illustrate the rich structure that lies within the original observation by Galileo. Furthermore, all of the methods used are easily accessible to any undergraduate student. Hence, some of the questions posed below may provide wonderful avenues for further exploration. Below are some natural questions that have risen from this study.

(1) Find other examples of alternating Galileo sequences.
(2) Explore the physical interpretations (if any) of ordinary and generalized alternating Galileo sequences.
(3) Study other generalizations of Galileo sequences.

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