FARKAS' LEMMA AND MULTILINEAR FORMS

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Abstract. In this note, we give a simple counterexample to a version of Farkas' Lemma for multilinear forms, and provide an elementary proof of a positive result for a special case. We also mention some remaining open problems.

The classical Farkas' Lemma states that if A_1, A_2, $\ldots$, A_k and A are real linear forms with the property that $A_i(v) \geq 0$ for all $i = 1, \ldots, k$ implies that $A(v) \geq 0$, then in fact $A = \sum_{i=1}^{k} \alpha_i A_i$, where $\alpha_i \geq 0$ for each $i = 1, \ldots, k$.

Numerous papers [1, 2, 3] have been published concerning novel proofs or generalizations of this useful result. In this paper, we consider an extension to multilinear forms. We give a positive result and pose some open problems, but Farkas' Lemma cannot be extended to multilinear forms in general, as Example 1 shows.

Example 1. Let $A, B, C\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the bilinear forms given by

$$\begin{aligned} A(x,y) &= (2x_1 + 3x_2)y_1 + (x_1 + 2x_2)y_2 \\ B(x,y) &= (3x_1 + x_2)y_1 + (2x_1 + x_2)y_2 \\ C(x,y) &= (2x_1 + x_2)y_1 + (x_1 + x_2)y_2. \end{aligned}$$

We claim that whenever $A(x,y) \geq 0$ and $B(x,y) \geq 0$, then $C(x,y) \geq 0$, but that C is linearly independent of A and B. To see this, first note that if $x = (0,0)$, then $A(x,y) = B(x,y) = C(x,y) = 0$ for any y. Thus, from here on we assume that $x \neq (0,0)$. Now, for a given x, let A_x, B_x, and C_x denote the linear forms obtained by "fixing" x (i.e. $A_x(y) = A(x,y)$ etc.). In order that C_x be a linear combination of A_x and B_x, say $C_x = \alpha_x A_x + \beta_x B_x$, α_x and β_x must be solutions of

$$\begin{pmatrix} 2x_1 + 3x_2 & 3x_1 + x_2 \\ x_1 + 2x_2 & 2x_1 + x_2 \end{pmatrix} \begin{pmatrix} \alpha_x \\ \beta_x \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}.$$

But the determinant of the matrix on the left is $x_1^2 + x_1x_2 + x_2^2$, which is strictly positive. Hence, for each x, α_x and β_x are uniquely determined and given by

$$\begin{aligned} \begin{pmatrix} \alpha_x \\ \beta_x \end{pmatrix} &= \frac{1}{x_1^2 + x_1x_2 + x_2^2} \begin{pmatrix} 2x_1 + x_2 & -3x_1 - x_2 \\ -x_1 - 2x_2 & 2x_1 + 3x_2 \end{pmatrix} \begin{pmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_1^2}{x_1^2+x_1x_2+x_2^2} \\ \frac{x_2^2}{x_1^2+x_1x_2+x_2^2} \end{pmatrix}. \end{aligned}$$

Now, if C were a linear combination of A and B, α_x, and β_x would necessarily be constant functions of x, which they clearly are not. Moreover, notice that α_x and β_x are nonnegative for each $x (x \neq 0)$. Therefore, if $A(x, y) \geq 0$ and $B(x, y) \geq 0$, then $C(x, y) = \alpha_x A(x, y) + \beta_x B(x, y) \geq 0$. So we see that A, B and C satisfy the hypothesis of Farkas' Lemma, even though C is linearly independent of A and B.

It turns out that Farkas' Lemma is true for multilinear forms in the special case when $k = 1$. We make use of Theorem 1 below, found in [4]. We note here that the strength of the following results is that the spaces are not assumed to be finite dimensional.

Theorem 1. Let A and B be two n-linear forms on the product $E_1 \times \ldots \times E_n$ of n vector spaces, with $A^{-1}(0) \subseteq B^{-1}(0)$. Then $B = \alpha A$ for some $\alpha \in \mathbb{K}$.

Theorem 2. Let A and B be two n-linear forms on the product $E_1 \times \cdots \times E_n$ of n vector spaces. If $A^{-1}[0, \infty) \subseteq B^{-1}[0, \infty)$, then $B = \alpha A$ for some $\alpha \geq 0$.

Proof. Let us first assume $B \neq 0$. Suppose that for some vector v, that $A(v) = 0$ and $B(v) \neq 0$. By our assumption, $B(v) > 0$. But then $A(-v) = 0$ and $B(-v) < 0$, contradicting the hypothesis. Hence, it must be that $A^{-1}(0) \subseteq B^{-1}(0)$. Theorem 1 then gives us that $A = \alpha B$ for some α. However, the hypothesis necessitates that $\alpha \geq 0$. If $B = 0$, it is clear by the hypothesis that $A = 0$, so we may choose $\alpha = 0$.

We mention now some related open problems. Note that Example 1 relied on the fact that a polynomial in two real variables can be strictly positive. This leaves open the possibility of a positive result for n-linear forms where n is odd. Furthermore, the forms in the counterexample are not symmetric, and the restriction of all forms involved to symmetric ones also warrants investigation.

References

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