PASTING LEMMAS FOR g-CONTINUOUS FUNCTIONS

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Abstract. The Pasting Lemma for continuous functions plays a key role in algebraic topology. Several mathematicians have established pasting lemmas for some stronger and weaker forms of continuous functions. In this paper we prove pasting lemmas for rg-continuous, gp-continuous, gc-irresolute, and gpr-continuous functions.

1. Introduction and Preliminaries. The Pasting Lemma for continuous functions has applications in algebraic topology. The continuous functions defined on closed sets of a locally finite covering of a topological space can be pasted to form a continuous function on the whole space. In this paper we establish the pasting lemma for rg-continuous [14], gc-irresolute [5], and gp-continuous [3] functions.

Throughout the paper, (X, τ) is a topological space on which no separation axiom is assumed unless explicitly stated. Let A be a subset of X. Then A is

(i) preopen [12] if $A \subseteq Int(Cl(A))$ and preclosed if $Cl(Int(A) \subseteq A$.
(ii) semi-open [10] if $A \subseteq Cl(Int(A))$ and semi-closed if $Int(Cl(A)) \subseteq A$.
(iii) regular open [15] if $A = Int(ClA))$ and regular closed if $A = Cl(Int(A))$.
(iv) generalized closed [11] (briefly g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open
(v) regular generalized closed [14] (briefly rg-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open.
(vi) generalized preclosed [4] (briefly gp-closed) if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
(vii) generalized pre-regular closed [7] (briefly gpr-closed) if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.

The complement of a g-closed set is g-open. Analogously, the concepts rg-open set, gp-open set, and gpr-open set will be defined.

Let $f: X \to Y$. Then f is

(i) g-continuous [5] if $f^{-1}(V)$ is g-closed for every closed set V of Y.
(ii) rg-continuous [14] if $f^{-1}(V)$ is rg-closed for every closed set V of Y.
(iii) semi-continuous [10] if $f^{-1}(V)$ is semi-closed for every closed set V of Y.
(iv) gp-continuous [3] if $f^{-1}(V)$ is gp-closed for every closed set V of Y.
(v) gpr-continuous [7] if $f^{-1}(V)$ is gpr-closed for every closed set V of Y.
(vi) gc-irresolute [5] if the inverse image of a g-closed set in Y is g-closed in X.

(vii) gp-irresolute [3] if the inverse image of a gp-closed set in Y is gp-closed in X.

A collection $\{A_\alpha : \alpha \in \Omega\}$ of subsets of a space X is locally finite [13] if every point of X has a neighborhood that intersects only finitely many members of $\{A_\alpha : \alpha \in \Omega\}$.

The following theorems and propositions will be useful in the sequel.

Proposition 1.1 [13]. Let $\{A_\alpha : \alpha \in \Omega\}$ be a locally finite collection of subsets of a space X. Then $Cl(\cup A_\alpha) = \cup Cl(A_\alpha)$.

Proposition 1.2 [11]. Suppose $B \subseteq A \subseteq X$, B is g-closed relative to A, and A is a g-closed subset of X. Then B is g-closed relative to X.

Proposition 1.3 [8]. Let $A \subseteq Y \subseteq X$. Then

(a) If Y is open in X and A is gpr-closed in X, then A is gpr-closed in Y and
(b) If Y is open and preclosed in X and A is gpr-closed in Y, then A is gpr-closed in X.

Proposition 1.4 [3]. Let $F \subseteq A \subseteq X$, where A is open and gp-closed in X. If F is gp-closed in A, then F is gp-closed in X.

Proposition 1.5 [1]. The union of two gpr-closed sets is gpr-closed if at least one of them is semi-closed.

Proposition 1.6 [6]. For a topological space X the following are equivalent.

(a) X is submaximal.
(b) $Cl(A) = pCl(A)$ for every subset A of X, where $pCl(A)$ is the preclosure of A.

Proposition 1.7 [5]. Let $X = A \cup B$ be a topological space with topology τ and Y be a topological space with topology σ. Let $f\colon (A, \tau|_A) \to (Y, \sigma)$ and $g\colon (B, \tau|_B) \to (Y, \sigma)$ be g-continuous maps such that $f(x) = g(x)$ for every $x \in A \cap B$. Suppose A and B are g-closed sets in X. Then the function $h\colon (X, \tau) \to (Y, \sigma)$, defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is g-continuous.

Proposition 1.8 [8]. Let $X = A \cup B$ be a topological space with topology τ and Y be a topological space with topology σ. Let the family of all gpr-open sets in (X, τ) be closed under finite intersections and let $f\colon (A, \tau|_A) \to (Y, \sigma)$ and $g\colon (B, \tau|_B) \to (Y, \sigma)$ be gpr-continuous maps such that $f(x) = g(x)$ for every $x \in A \cap B$. Suppose A and B are open and preclosed in X. Then the function $h\colon (X, \tau) \to (Y, \sigma)$ defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is gpr-continuous.

Proposition 1.9 [2]. If A is semi-closed, then $pCl(A \cup B) = pCl(A) \cup pCl(B)$.

Proposition 1.10 [3]. In a submaximal space, every gp-closed set is g-closed.

Arbitrary union of g-closed (resp. rg-closed) sets is not g-closed (resp. rg-closed). However, we will prove that the union of a locally finite collection of g-closed sets (resp. rg-closed) is g-closed (resp. rg-closed).

2. Pasting Lemmas. In this section we prove that the union of g-closed sets from a locally finite family of g-closed sets is g-closed, the union of gp-closed sets from a locally finite family of gp-closed sets in a submaximal space is gp-closed, and we use them to generalize the pasting lemma.

Theorem 2.1. If $\{A_\alpha : \alpha \in \Omega\}$ is a locally finite family of g-closed (resp. rg-closed) sets, then $\cup A_\alpha$ is g-closed (resp. rg-closed).

Proof. Let $\{A_\alpha\}$ be a locally finite collection of g-closed (resp. rg-closed) sets in X and let $\cup A_\alpha \subseteq U$, where U is open (resp. regular open) in X. Then $A_\alpha \subseteq U$ implies $Cl(A_\alpha) \subseteq U$. This implies $\cup Cl(A_\alpha) \subseteq U$. By Proposition 1.1, $Cl(\cup A_\alpha) \subseteq U$. Therefore, $\cup A_\alpha$ is g-closed (resp. rg-closed).

Corollary 2.2. If $\{A_\alpha : \alpha \in \Omega\}$ is a locally finite family of gp-closed sets of submaximal space X, then $\cup A_\alpha$ is gp-closed.

Proof. The corollary follows from Proposition 1.10 and Theorem 2.1.

Theorem 2.3. Let $X = \cup A_\alpha$ and let $\{A_\alpha : \alpha \in \Omega\}$ be a locally finite covering of g-closed sets. Let $f_\alpha : A_\alpha \to Y$ be g-continuous (resp. rg-continuous, resp. gc-irresolute) for all $\alpha \in \Omega$ such that $f_\alpha(x) = f_\beta(x)$ for all $x \in A_\alpha \cap A_\beta$. Define $f(x) = f_\alpha(x)$ for $x \in A_\alpha$. Then f is g-continuous (resp. rg-continuous, resp. gc-irresolute).

Proof. Let F be closed (resp. g-closed) in Y. Then $f^{-1}(F) = \cup f_\alpha^{-1}(F)$. Since f_α is g-continuous in A_α, $f_\alpha^{-1}(F)$ is g-closed in A_α for all α. By Proposition 1.2, $f_\alpha^{-1}(F)$ is g-closed in X for all α. Since $f_\alpha^{-1}(F) \subseteq A_\alpha$ for all α and since $\{A_\alpha : \alpha \in \Omega\}$ is locally finite, $\{f_\alpha^{-1}(F) : \alpha \in \Omega\}$ is locally finite. Then $\cup f_\alpha^{-1}(F)$ is g-closed (resp. rg-closed) in X.

Theorem 2.4. Let $X = A \cup B$, where A and B are both open and preclosed in X. Let $f : A \to Y$ and $g : B \to Y$ be gpr-continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Define $h : X \to Y$ such that $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$. Furthermore, if f is semi-continuous (or) g is semi-continuous, then h is gpr-continuous.

Proof. Let F be closed in Y. Then $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$, where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. By Proposition 1.3, C is gpr-closed in X. Similarly, D is gpr-closed in X. Since f is semi-continuous, $f^{-1}(F)$ is semi-closed. By Theorem 1.5, $C \cup D$ is gpr-closed in X. Therefore, $h^{-1}(F)$ is gpr-closed in X. Hence, h is gpr-continuous.

Theorem 2.5. Let $\{A_\alpha : \alpha \in \Omega\}$ be a locally finite collection of subsets of a submaximal space X. Then $pCl(\cup A_\alpha) = \cup pCl(A_\alpha)$.

Proof. The theorem follows from Proposition 1.1 and Proposition 1.6.

Theorem 2.6. Let X be a submaximal space and let $\{A_\alpha : \alpha \in \Omega\}$ be a locally finite covering of subsets of X such that each A_α is gp-closed in X. Let $f_\alpha : A_\alpha \to Y$ be gp-continuous for all $\alpha \in \Omega$ such that $f_\alpha(x) = f_\beta(x)$ for all $x \in A_\alpha \cap A_\beta$. Define $f(x) = f_\alpha(x)$ for $x \in A_\alpha$. Then f is g-continuous.

Proof. Let F be closed in Y. Since f_α is gp-continuous in A_α, $f_\alpha^{-1}(F)$ is gp-closed in A_α for all α. Since each A_α is submaximal, $f_\alpha^{-1}(F)$ is g-closed in A_α. By Proposition 1.2, $f_\alpha^{-1}(F)$ is g-closed in X for all α and hence, $f_\alpha : A_\alpha \to Y$ is g-continuous for each α. Since $f_\alpha^{-1}(F) \subseteq A_\alpha$ for all α and since $\{A_\alpha : \alpha \in \Omega\}$ is locally finite in X, $\{f_\alpha^{-1}(F) : \alpha \in \Omega\}$ is a locally finite family of g-closed sets in X. Then by Theorem 2.3, f is g-continuous. Again since $f^{-1}(F) = \cup f_\alpha^{-1}(F)$, by Theorem 2.1, $\cup f_\alpha^{-1}(F)$ is g-closed. This shows that f is g-continuous.

Lemma 2.7. The union of two gp-closed sets is gp-closed if at least one of them is semi-closed.

Proof. Let $A \cup B \subseteq U$, where U is open and A and B are gp-closed. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are gp-closed, $pCl(A) \subseteq U$ and $pCl(B) \subseteq U$. Since A is semi-closed, by Proposition 1.9, $pCl(A \cup B) \subseteq U$. Therefore, $A \cup B$ is gp-closed.

Theorem 2.8. Let $X = A \cup B$, where A and B are both open and gp-closed. Let $f : A \to Y$ and $g : B \to Y$ be gp-continuous (resp. gp-irresolute) functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Define $h : X \to Y$ such that $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$. Furthermore, if f is semi-continuous or g is semi-continuous, then h is gp-continuous (resp. gp-irresolute).

Proof. Let F be a closed (resp. gp-closed) set in Y. Then $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$, where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. By Proposition 1.4, C is gp-closed in X. Similarly, D is gp-closed in X. Since f is semi-continuous, $f^{-1}(F)$ is semi-closed. Therefore, by using Lemma 2.7, $h^{-1}(F)$ is gp-closed in X.

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References

1. M. Anitha and P. Thangavelu, "A Note on gpr-closed Sets in Topology," *Journal of Indian Acad. Math.*, 28 (2006), 475–481.

2. M. Anitha and P. Thangavelu, "Generalized Closed Sets in Submaximal Spaces," *Antarctica J. Math.*, 4 (2007), 99–108.

3. I. A. Rani, K. Balachandran, and J. Dontchev, *Topology Atlas*, preprint, `http://www.unipissing.ca\topology\p\a\a\j\14.html`.

4. K. Balachandran and A. Rani, "On Generalized Preclosed Sets," preprint.

5. K. Balachandran, P. Sundaram, and H. Maki, "On Generalized Continuous Maps in Topological Spaces," *Mem. Fac. Sci. Kochi Univ. (Math.)*, 12 (1991), 5–13.

6. J. Dontchev, "On Sub Maximal Spaces," *Tamkang J. Math.*, 26 (1995), 243–250.

7. Y. Gnanambal, "On Generalized Preregular Closed Sets in Topological Space," *Indian J. Pure Appl. Math.*, 28 (1997), 351–360.

8. Y. Gnanambal and K. Balachandran, "On gpr-continuous Functions in Topological Spaces," *Indian J. Pure Appl. Math.*, 30 (1999), 581–593.

9. J. H. Park, Y. B. Park, B. Y. Lee, "On gp-closed Sets and Pre-gp-continuous Functions," *Indian J. Pure Appl. Math.*, 33 (2002), 3–12.

10. N. Levine, "Semi-open Sets and Semi-continuity Topological Spaces," *Amer. Math. Monthly*, 70 (1963), 36–41.

11. N. Levine, "Generalized Closed Sets in Topology," *Rend. Circ. Mat. Palermo - Serie II - Tomo XIX - Anno*, (1970), 89–96.

12. A. S. Mashhour, M. E. A. E. Monsef, and S. N. E. Deeb, "On Pre-continuous and Weak Pre-continuous Mappings," *Proc. Math. Phys. Soc. Egyp.*, 53 (1982), 47–53.

13. J. R. Munkers, *Topology*, second edition, PHI (2004).

14. N. Palaniappan and K. C. Rao, "Regular Generalized Closed Sets," *Kyungpook Math. J.*, 33 (1993), 211–219.

15. M. H. Stone, "Applications of the Theory of Boolean Rings to the General Topology," *Trans. A. M. S.*, 41 (1937), 375–481.

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