AN n-CELL IN $\mathbb{R}^{n+1}$ THAT IS NOT THE ATTRACTOR OF ANY IFS ON $\mathbb{R}^{n+1}$

Manuel J. Sanders

Abstract. Crovisier and Rams [2] recently constructed an embedded Cantor set in $\mathbb{R}$ and showed that it could not be realized as an attractor of any iterated function system (IFS) using measure-theoretic properties. Also, an example of a locally connected continuum in $\mathbb{R}^2$ which is not the attractor of any IFS on $\mathbb{R}^2$ is constructed in a work of Kwieciński [6]. Kwieciński points out that a variation on his main construction provides an arc in $\mathbb{R}^2$ which is not the attractor of any IFS either. In this work, for each $n \geq 1$, we construct an n-cell in $\mathbb{R}^{n+1}$ and show that this n-cell cannot be the attractor of any IFS on $\mathbb{R}^{n+1}$. The $n = 1$ case reaffirms the result observed by Kwieciński.

1. Introduction. If $X = X_d$ is a metric space, a contraction map on X is a function $f\colon X \to X$ with the feature that $d(f(x), f(y)) \leq r \cdot d(x, y)$ for each $x, y \in X$, where $0 \leq r < 1$. It is easily verified that such a function is necessarily continuous, so that the word map is used in the usual way. A (hyperbolic) iterated function system (IFS) on X consists of a complete metric space X together with a finite set of contraction maps on X. There is a unique compact subset of X associated with a given IFS called the attractor of the IFS. Barnsley [1] popularized the notion of an iterated function system by revealing its ability to encode a subset of the ambient space in the contraction maps of the IFS.

Mathematically, there remain open questions about IFSs and their attractors. While it is possible to approximate any compact subset in the space X by an attractor of some IFS [1], the question as to which compacta can be realized as attractors of IFSs remains elusive. Earlier works include [3, 6]. Kwieciński [6] gives an example of a locally connected continuum which cannot be realized as an attractor of any IFS in $\mathbb{R}^2$. A variation on his main argument provides an example of an arc which is not an attractor of any IFS as well. While Hata [5] showed that a connected attractor must be locally connected, he went on to pose the question of whether or not each locally connected continuum in $\mathbb{R}^n$ could be realized as the fixed point set of a finite collection of weak contractions. Kwieciński's result relates to this question by showing that, indeed, there are locally connected continua which are not realizable as attractors of IFSs.

In this work, we shall develop techniques to point out that many arcs in $\mathbb{R}^n$ are not attractors of any IFS ($n \geq 2$) by exploiting a characteristic pertaining to a particular embedding of $[0, 1]$ into $\mathbb{R}^n$. Kwieciński's example will be reaffirmed by this characteristic. More to the point, we will be able to produce many arcs that cannot be realized as attractors, each leading to embedded n-cells that also cannot be realized as attractors. On a side note, we shall also state a sufficient condition for an arc to be realizable as an attractor of some IFS.

Throughout, we will write $\{X; f_1, f_2, \dots f_k\}$, where X is a complete metric space and f_i is a contraction map on X to denote an iterated function system. The unique attractor $A \subset X$ associated with this IFS is invariant in the sense that

$$A = f_1(A) \cup f_2(A) \cup \cdots \cup f_k(A)$$

and, indeed, is uniquely characterized by this equation [1].

2. Length of an Arc and Preliminary Matters. We shall let I denote the interval $[0, 1]$ from here forward. If $e\colon I \to \mathbb{R}^n$ is an embedding, $e(I)$ is called a curve (or more usually here, an arc) [4]. The *length of the arc* $e(I)$ is defined as follows: Let $P = \{x_0 = 0 < x_1 < x_2 < \cdots < x_k = 1\}$ be a partition of I. Let $\mathcal{L}_e(P) = \sum_{i=1}^{k} |e(x_i) - e(x_{i-1})|$, where $|a - b|$ denotes the usual Euclidean distance from a to b in $\mathbb{R}^n$.

Then, $\mathcal{L}_e = \sup\{\mathcal{L}_e(P) : P \text{ is a partition of } I\}$ is called the *variation of e on I*. Note that the variation of e on I has only to do with the image of I under e, that is on the arc itself as a point set. Therefore, if e' is any other embedding of I so that $e(I) = e'(I)$, $\mathcal{L}_e = \mathcal{L}_{e'}$. The length of the arc $e(I)$ is defined to be the variation of e on I unambiguously in this manner. For a given arc A in $\mathbb{R}^n$, it is meant to be understood that the arc arises from some embedding $e\colon I \to \mathbb{R}^n$. We mention this embedding only if it is necessary or convenient. Because of this arrangement, we may consider an arc A to be ordered naturally (in one of two manners) and speak of an *ordered arc*. Moreover, we shall frequently make reference to the endpoints of an arc in the natural way. Thus, if A is an arc with endpoints $a < b$, we freely talk about a point c of A satisfying $a < c < b$. Notationally, we will write $\mathcal{L}(A) = \mathcal{L}_a^b$ to denote the length of an arc A with endpoints $a, b \in \mathbb{R}^n$. (So, $\mathcal{L}_a^b = \mathcal{L}_b^a$.) Related to this convention, we will define $\mathcal{L}_a^a = 0$ so that the length of a point is zero.

Elementary Properties Involving the Length of an Arc The following several observations are easy to check: Let A be an arc in $\mathbb{R}^n$ with endpoints $a < b$. Then,

- $\mathcal{L}(A) > 0$,
- If $a < c < b$, then $\mathcal{L}_a^c + \mathcal{L}_c^b = \mathcal{L}_a^b$,
- If C is a subarc of A, then $\mathcal{L}(C) \leq \mathcal{L}(A)$, and
- If $\mathcal{L}(A) < \infty$, the function $v\colon A \to [0, \infty)$ defined by $v(x) = \mathcal{L}_a^x$ is strictly increasing.

Along the same lines, we state the following propositions whose proofs are omitted as well.

Proposition 2.1. For $n \geq 1$, let $A \subset \mathbb{R}^n$ be an ordered arc with endpoints $a < b$. Suppose $\mathcal{L}_a^b = \mathcal{L}(A) < \infty$. Then $v: A \to [0, \infty)$ defined by $v(x) = \mathcal{L}_a^x$ is continuous.

Comment. If X, Y are metric spaces with metrics d_X and d_Y, respectively, a map $f: X \to Y$ is said to be a Lipschitz map provided there is a $k \in \mathbb{R}$ so that $d_Y(f(x), f(y)) \leq k \cdot d_X(x, y)$ for all $x, y \in X$. Here, k is called a Lipschitz constant for f. It follows readily that every Lipschitz map is necessarily (uniformly) continuous. In this terminology, a contraction is a Lipschitz map with a Lipschitz constant less than 1.

3. Arcs of Finite Length are Attractors. The following theorem and corollary are proved in [9].

Theorem 3.1. For $n \geq 1$, let A be an arc in $\mathbb{R}^n$ and suppose $\mathcal{L}(A) < \infty$. Then A is the attractor of an IFS on $\mathbb{R}^n$.

Corollary 3.1. Let $A_1, A_2, A_3, \ldots, A_m$ be arcs in $\mathbb{R}^n$, each with finite variation. Then $\bigcup_{i=1}^m A_i$ is the attractor of an IFS on $\mathbb{R}^n$.

4. Working with Arc Length. To begin to establish the main ideas in this work, namely, constructing n-cells which are not attractors of any IFS on $\mathbb{R}^{n+1}$, we use the following several propositions, the proofs of which are left to the reader.

Proposition 4.1. Let A, C be arcs in $\mathbb{R}^n$. If $f: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with Lipschitz constant k so that $f(C) \subset A$, then $\mathcal{L}(f(C)) \leq k \cdot \mathcal{L}(C)$.

Comment. While there are no restrictions on the map f above such as requiring f to embed C in A, we note that $\mathcal{L}(f(C))$ still makes sense; $f(C)$ is compact and connected in A and thus is either a point or a subarc of A. The length of either is defined.

Proposition 4.2. Let A be an arc in $\mathbb{R}^n$ with endpoints $a < b$. Suppose $\{c_m\}$ is a sequence of points of A satisfying

1. $c_0 = a$
2. $c_m \to b$ as $m \to \infty$.

Then for $j \in \mathbb{N}$, if B_j denotes the possibly degenerate subarc of A with endpoints c_{j-1} and c_j, then $\mathcal{L}_a^b = \mathcal{L}(A) \leq \sum_{j=1}^{\infty} \mathcal{L}(B_j)$.

With the above notions in place, we are ready to present a theorem which, for each $n \geq 2$, provides an arc embedded in $\mathbb{R}^n$ which is the attractor of no IFS. While the same result is contained in the more general Theorem 5.1, the proof is prototypical of the arguments used in the more general setting of the proof of Theorem 5.1. The theorem will be stated first, then an example of an arc in $\mathbb{R}^2$ meeting the hypotheses of the theorem will be identified. The proof will follow the example.

Theorem 4.1. Let A be an arc in $\mathbb{R}^n$ with endpoints $a < b$. Suppose

1. $\mathcal{L}_x^y < \infty$ for all $x, y \in A$ with $x, y \neq b$, and
2. $\mathcal{L}_x^b = \infty$ for all $x \in A$ with $x \neq b$.

Then for any finite set of contractions $\{w_1, w_2, \dots, w_N\}$ on $\mathbb{R}^n$, A is not the attractor of the IFS $\{\mathbb{R}^n; w_1, w_2, \dots, w_N\}$.

Example 1. A Harmonic Spiral. The construction of this arc is based on the divergent harmonic series. We'll use the descriptive terms north, south, east, and west to label directions that correspond to directions in the plane in order to describe the arc. Start at the origin. Proceed 1 unit east. Turn north and proceed 1/2 unit. Turn west and proceed 1/3 unit. Turn south and proceed 1/4 unit. Turn east and proceed 1/5 unit. Continue on in this fashion. The desired arc spirals endlessly around a point that is related to the harmonic series.

Proof of Theorem 4.1. (By contradiction.) Assume that A is the attractor of some IFS $\{\mathbb{R}^n; w_1, w_2, \dots, w_N\}$. The first claim is that there exists a contraction w_J which is nonconstant on A so that $b \in w_J(A)$. To see this, note that there exists a contraction w_J such that $b \in w_J(A)$ and therefore, $w_J(A)$ must be the singleton $\{b\}$ if the claim is false. So, if the claim is false, we may assume without loss of generality that $b \notin w_i(A)$ if $i \neq J$ (else we have two functions that map A constantly to b and A would still be the attractor of the IFS obtained by removing one of these functions from the list of contractions). But then, A could be written as a union of two nonempty, disjoint closed sets, namely $\{b\} \cup \bigcup_{i \neq J} w_i(A)$. This violates the fact that A is connected and establishes the claim. (Thus in fact, we can assume that none of the contractions are constant on A, although we will not need this assumption again.) Relabel w_J as f for convenience.

Now, the claim is that b must be a fixed point of f. To see this, note that if $x \neq b$, then $f(x) \neq b$; there exists $y \neq b$ such that $f(y) \neq b$ as f is nonconstant on A and contains b in its image. If $f(x) = b$, the length of the arc with endpoints $f(x)$ and $f(y)$ would be infinite, whereas the length of the arc with endpoints x and y would be finite, contradicting Proposition 4.1. As b lies in the image of A under f, it must be that $f(b) = b$.

Let k denote the contractivity factor of f. The next claim is that $f^m(a) \neq b$ for any $m \in \mathbb{N}$. By definition, let $f^0(a) = a$. Note that $a \neq b$. Suppose $f(a) = b$. Then because f is nonconstant on A, there exists $c \in A$ so that $f(c) \neq b$. Note that $c \neq b$. Then by assumption, $\mathcal{L}_a^c < \infty$. But, $\mathcal{L}_{f(a)=b}^{f(c)\neq b} = \infty$. This contradicts Proposition 4.1. Hence, $f(a) \neq b$. Inductively, assume that for some positive integer r, $f^j(a) \neq b$ for every nonnegative integer $j \leq r$. If $f^{r+1}(a) = b$, then, $\mathcal{L}_{f^r(a)}^{f^{r+1}(a)=b} = \infty$ but $\mathcal{L}_{f^{r-1}(a)}^{f^r(a)} < \infty$. This leads to a contradiction with Proposition 4.1 again. Hence, it must be that $f^m(a) \neq b$ for every $m \in \mathbb{N}$ as desired.

Recapping,

1) $f^m(a) \in A$ since $f\colon A \to A$, and
2) $f^m(a) \to b$ as $m \to \infty$ (since $|f^m(a) - b)| \leq k^m \cdot |a - b|$ using the contractivity of f applied m times and the fact $0 \leq k < 1$, together with the result that b is a fixed-point of f.)

Now for $j \in \mathbb{N}$, let B_j denote the, possibly degenerate, subarc of A with endpoints $f^{j-1}(a)$ and $f^j(a)$. Then $\mathcal{L}(B_1) = \mathcal{L}_a^{f(a)} = L < \infty$ because B_1 has endpoints a and $f(a)$. Hence, $\mathcal{L}(B_2) = \mathcal{L}_{f(a)}^{f^2(a)} \leq k \cdot \mathcal{L}(B_1) = k \cdot L$ by Proposition 4.1. Furthermore, for any positive integer j, we may inductively determine that $\mathcal{L}(B_j) \leq k^{j-1} \cdot L$. Then in Proposition 4.2, take the sequence $\{c_m\}$ to be $\{f^m(a)\}$. The hypotheses of Proposition 4.2 are satisfied by 1) and 2) above so that $\mathcal{L}(A) \leq \sum_{i=1}^{\infty} \mathcal{L}(B_i)$. But then

$$\mathcal{L}(A) \leq \sum_{i=1}^{\infty} \mathcal{L}(B_i) \leq L + kL + k^2L + \cdots = L \cdot \sum_{i=1}^{\infty} k^i.$$

This last infinite series converges as $k < 1$. Hence, A is an arc of finite variation. This is a contradiction and it follows that A must not be the attractor of this IFS after all.

5. Producing n-cells in $\mathbb{R}^{\mathbf{n+1}}$ that are not Attractors of any IFS on $\mathbb{R}^{\mathbf{n+1}}$. The argument for producing n-cells that are not attractors in $\mathbb{R}^{n+1}$ is very similar to the proof for arcs given above.

Theorem 5.1. For each $n \geq 1$, there exists an n-cell in $\mathbb{R}^{n+1}$ which is not the attractor of any IFS on $\mathbb{R}^{n+1}$.

Proof. Theorem 4.1 verifies the $n = 1$ case. For $n > 1$, the proof will follow by contradiction. For $n > 1$, we consider the n-cell in $\mathbb{R}^{n+1}$ constructed as follows. Choose an arc A in $\mathbb{R}^2$ that meets the hypotheses of Theorem 4.1. Let $e\colon I \to \mathbb{R}^2$ be an embedding of A with the feature that $e(0) = a$ and $e(1) = b$. In $\mathbb{R}^{n+1}$, consider the subset $\{x \times [0, 1 - e^{-1}(x)]^{n-1} : x \in A\}$ of $A \times I^{n-1}$. Label this set X. The homeomorphism $e^{-1} : A \to I$ extends to a homeomorphism of X to $\{t \times [0, (1-t)]^{n-1} : t \in I\}$ exhibiting X as an n-cell embedded in $\mathbb{R}^{n+1}$. Let a_0 denote $a \times \{0\}^{n-1}$ and b_0 denote $b \times \{0\}^{n-1}$.

Let $\rho\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote projection onto $\mathbb{R}^2 \subset \mathbb{R}^{n+1}$. That is, let ρ be defined by $\rho(x_1, x_2, x_3, \ldots, x_{n+1}) = (x_1, x_2, 0, 0, \ldots, 0)$. Then $\rho(X) = A$ and we have

1. for $x \in X$, $\rho(x) = b_0 \Rightarrow x = b_0$, and
2. for any $x, y \in X$ with $x, y \neq b_0$, there exists a possibly degenerate arc with endpoints x, y so that $\mathcal{L}_x^y < \infty$.

Suppose X is the attractor of some IFS $\{\mathbb{R}^{n+1}; w_1, w_2, w_3, \dots w_N\}$. Then, X is uniquely characterized by the equation

$$X = \bigcup_{i=1,2,\dots,N} w_i(X). \tag{1}$$

We claim that there exists a contraction w_J which is nonconstant on X so that $b_0 \in w_J(X)$. To see this, note that there exists a contraction w_J such that $b_0 \in w_J(X)$ and therefore, w_J must be constant on X if the claim is false. So, if the claim is false, we may assume without loss of generality that $b_0 \notin w_i(X)$, if $i \neq J$ (else we have two contractions with the same image in X and (by (1)) we may remove one of these from the list of contractions to obtain an IFS with the same attractor X). But if so, then X could be written as a union of two nonempty, disjoint closed sets, namely $\{b_0\} \cup \bigcup_{i \neq J} w_i(X)$. This violates the connectedness of X and the claim is verified. Relabel w_J as f for convenience. Let k denote the contractivity factor of f. Then the contractivity factor of ρf is less than or equal to k as well since ρ is a projection.

Now, the next claim is that b_0 must be a fixed point of f. To see this, note that if $x \in X$ with $x \neq b_0$, then $f(x) \neq b_0$; there exists $y \neq b_0$ such that $f(y) \neq b_0$ as f is nonconstant on X and contains b_0 in its image. Choose an arc C in X with endpoints x and y so that $\mathcal{L}_x^y < \infty$ by 2. above. Then as ρf is a contraction on $\mathbb{R}^{n+1}$ and maps C into the arc A and $\mathcal{L}_x^y < \infty$ and yet $\mathcal{L}_{\rho f(y) \neq b_0}^{\rho f(x) = b_0} = \infty$, we have a contradiction to Proposition 4.1. As $f(b_0)$ lies in the image of X under f, it must be that $f(b_0) = b_0$.

The next claim is that $f^m(a_0) \neq b_0$ for any $m \in \mathbb{N}$. By definition, let $f^0(a_0) = a_0$. Note that $a_0 \neq b_0$. Suppose $f(a_0) = b_0$. Then because f is nonconstant on X, there exists $c \in X$ so that $f(c) \neq b_0$. Note that $c \neq b_0$. Choose an arc C_1 in X with endpoints a_0 and c so that $\mathcal{L}_{a_0}^c < \infty$ by 2. above. Then $\mathcal{L}_{a_0}^c < \infty$, but $\mathcal{L}_{\rho f(a_0) = b_0}^{\rho f(c) \neq b_0} = \infty$. This contradicts Proposition 4.1. Hence, $f(a_0) \neq b_0$. Inductively, assume that for some positive integer r, $f^j(a_0) \neq b_0$ for every nonnegative integer $j \leq r$. Choose an arc C_r in X with endpoints $f^{r-1}(a_0)$ and $f^r(a_0)$ so that $\mathcal{L}_{f^{r-1}(a_0)}^{f^r(a_0)} < \infty$ by 2. above again. If $f^{r+1}(a_0) = b_0$, then $\mathcal{L}_{\rho f^r(a_0)}^{\rho f^{r+1}(a_0) = b_0} = \infty$, but $\mathcal{L}_{f^{r-1}(a_0)}^{f^r(a_0)} < \infty$. This leads to a contradiction with Proposition 4.1 again. Hence, it must be that $f^m(a_0) \neq b_0$ for every $m \in \mathbb{N}$ as desired.

Recapping, we have

1. $f^m(a_0) \in X \Rightarrow \rho f^m(a_0) \in A$ as $\rho f^m(X) \subset A$
2. $\rho f^m(a_0) \to b_0$ as $m \to \infty$ since $|\rho f^m(a_0) - b_0| \leq k^m \cdot |a_0 - b_0|$ (using the contractivity factor of ρf applied m times and the fact $0 \leq k < 1$, together with the result that b_0 is a fixed-point of f and hence, ρf.)

Now for $j \in \mathbb{N}$, let B_j denote the possibly degenerate subarc of A with endpoints $(\rho f)^{j-1}(a_0)$ and $(\rho f)^j(a_0)$. (So, the endpoints of B_1 are a_0

and $\rho f(a_0)$, the endpoints of B_2 are $\rho f(a_0)$ and $\rho f \rho f(a_0)$, the endpoints of B_3 are $\rho f \rho f(a_0)$ and $\rho f \rho f \rho f(a_0)$, etc.) Note that $(\rho f)^j$ has contractivity factor k^j. Then $\mathcal{L}(B_1) = \mathcal{L}_{a_0}^{\rho f(a_0)} = L < \infty$ because $a_0, \rho f(a_0) \in A$ and $a_0, \rho f(a_0) \neq b_0$. Hence, $\mathcal{L}(B_2) = \mathcal{L}_{\rho f(a_0)}^{(\rho f)^2(a_0)} \leq k \cdot \mathcal{L}(B_1) = k \cdot L$ by Proposition 4.1. Furthermore, for any positive integer j, we may inductively determine that $\mathcal{L}(B_j) \leq k^{j-1} \cdot L$. Then in Proposition 4.2, take the sequence $\{c_m\}$ to be $\{(\rho f)^m(a_0)\}$. The hypotheses of Proposition 4.2 are satisfied by the bulleted items above so that $\mathcal{L}(A) \leq \sum_{i=1}^{\infty} \mathcal{L}(B_i)$. But then,

$$\mathcal{L}(A) \leq \sum_{i=1}^{\infty} \mathcal{L}(B_i) \leq L + kL + k^2L + \cdots = L \cdot \sum_{i=1}^{\infty} k^i.$$

This last infinite series converges as $k < 1$. Hence, A is an arc of finite variation. From this contradiction, it follows that X must not be the attractor of the given IFS after all.

6. Open Questions. Upon these developments, several natural questions arise.

- What if the hypotheses of Theorem 5.1 are modified so that a single "bad" point occurs as an interior point of the n-cell? Could this n-cell be an attractor of some IFS in $\mathbb{R}^{n+1}$?
- Could an n-cell with a finite number of "bad" points be an attractor?
- Since an arc with infinite variation between each pair of distinct points (i.e., where all points are "bad") can be realized as an attractor of an IFS (consider the von Koch arc for instance), is it possible to construct an arc with this type of feature that must be an attractor of no IFS? Suggestion from the referee: The contraction cannot increase box dimension/Hausdorff dimension/packing dimension of the set. Construction of a von Koch-type curve with some sort of dimension monotonically increasing 'from left to right' would produce a similar kind of curve as that in Theorem 4.1 which would contain an exceptional point.
- If a set $X \subset \mathbb{R}^n$ is the attractor of an IFS on $\mathbb{R}^n$, then it follows that $X \times I$ is an attractor of an IFS on $\mathbb{R}^{n+1}$. If X is **not** an attractor of any IFS on $\mathbb{R}^n$, is it possible that $X \times I$ could be an attractor on $\mathbb{R}^{n+1}$?

References

1. M. Barnsley, *Fractals Everywhere*, Academic Press, Inc., Boston, MA, 1988.

2. S. Crovisier and M. Rams, "IFS Attractors and Cantor Sets," *Topology Appl.*, 153 (2006), 1849–1859.

3. P. F. Duvall and L. S. Husch, "Attractors of Iterated Function Systems," *Proc. Amer. Math. Soc.*, 116 (1992), 279–284.

4. K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, New York, NY, 1985.

5. M. Hata, "On the Structure of Self-Similar Sets," *Japan J. Appl. Math*, 2 (1985), 381–414.

6. M. Kwieciński, "A Locally Connected Continuum Which is Not an IFS Attractor," *Bull. Polish Acad. Sci.*, 47 (1999), 128–132.

7. J. Luukkainen and J. Vaisala, "Elements of Lipschitz Topology," *Annales Academiae Scientarium Fennicae*, 3 (1977), 85–122.

8. J. R. Munkres, *Topology: A First Course*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

9. M. J. Sanders, "Nonattractors of Iterated Function Systems," *Texas Project NexT e-Journal:* `http://www.cs.southwestern.edu/txcmj/txnext.htm`, 1 (2003), 1–9.

Mathematics Subject Classification (2000): 54H20, 37B45

Manuel J. Sanders
Department of Science and Mathematics
University of South Carolina Beaufort
Bluffton, SC 29909
email: mjsander@uscb.edu