# p-COLORING CLASSES OF TORUS KNOTS

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**Abstract.** We classify by elementary methods the $p$-colorability of torus knots, and prove that every $p$-colorable torus knot has exactly one nontrivial $p$-coloring class. As a consequence, we note that the two-fold branched cyclic cover of a torus knot complement has cyclic first homology group.

**1. Introduction.** Our first result is a theorem specifically determining the $p$-colorability of any $(m, n)$ torus knot. It has been previously shown that a $(m, m-1)$ torus knot is always $p$-colorable for $p$ equal to $m$ or $m-1$ depending on which is odd [6, 13]. Another proven result is that a $(2, n)$ torus knot is always $p$-colorable for $p$ equal to $n$ and a $(3, n)$ torus knot is always 3-colorable if $n$ is even [13]. A result similar to ours was also stated as a lemma without proof in [3].

Our second result shows that any $p$-colorable $(m, n)$ torus knot has only one nontrivial $p$-coloring class. A general result investigating colorings of torus knots by finite Alexander quandles appears in [2]. Our result is a special instance of this result; however, we present a proof using only elementary techniques. $p$-coloring classes have also previously been investigated in relationship to pretzel knots in [4]. An immediate corollary of this result is that any nontrivial $p$-coloring of the standard braid representation of $T_{m,n}$ must use all $p$ colors. Distribution of colors in $p$-colorings of knots has been previously investigated with the Harary-Kauffman Conjecture, which examines the distribution of colors in a $p$-coloring of an alternating knot with prime determinant. Asaeda, Przytcki, and Sikora prove the Harary-Kauffman Conjecture is true for pretzel knots and Montesinos knots in [1]. Another immediate corollary is that the first homology groups of certain branched cyclic covers of torus knot complements [5, 11] are cyclic.

**2. Notation.** Given a prime $p > 2$ and a projection of a knot $K$ with strands $s_1, s_2, \ldots, s_r$, a *p-coloring* is an assignment $c_1, c_2, \ldots, c_r$ of elements of $\mathbb{Z}_p$ to the strands of the projection that satisfies the condition that at each crossing, where $s_i, s_j$ are the undercrossing strands and $s_k$ is the overcrossing strand, $c_i + c_j - 2c_k = 0 \bmod p$. A $p$-coloring is said to be *nontrivial* if at least two distinct "colors" in $\mathbb{Z}_p$ are used. $p$-colorability is invariant under Reidemeister moves and is thus a knot invariant, so a knot $K$ is $p$-colorable if its projections admit nontrivial $p$-colorings.

Equivalently, a knot $K$ is $p$-colorable if there exists an onto homomorphism from the knot group $\pi_1(S^3 - K)$ of $K$ to the dihedral group

$$D_{2p} = \langle a, b \mid a^2 = 1, b^p = 1, abab = 1 \rangle.$$

The knot group $G$ can be expressed via the *Wirtinger presentation* as follows. Given a projection of $K$, define loops $x_1, x_2, \ldots, x_r$ around the strands $s_1, s_2, \ldots, s_r$, respectively, following the right hand rule. At each crossing, where $s_i$ terminates, $s_j$ originates, and $s_k$ is the overcrossing strand, we have a relation $R_j$ that is either of the form $x_j = x_k^{-1} x_i x_k$ or $x_j = x_k x_i x_k^{-1}$, depending on whether the sign of the crossing is positive or negative [12]. With this notation, the Wirtinger presentation for the knot group of $K$ is:

$$\pi_1(S^3 - K) = \langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_r \rangle.$$

It is a simple exercise to prove that an assignment of colors $c_1, c_2, \ldots, c_r \in \mathbb{Z}_p$ is a proper $p$-coloring of a projection of $K$ if and only if the map $\theta\colon \pi_1(S^3 - K) \to D_{2p}$ defined by $\theta(x_i) = ab^{c(s_i)}$ is an onto homomorphism [8].

Two $p$-colorings $c_1, c_2, \ldots, c_r$ and $d_1, d_2, \ldots, d_r$ of a projection of $K$ are said to be *equivalent*, or in the same *p-coloring class* if for all $1 \le i, j \le r$, $c_i = c_j$ if and only if $d_i = d_j$; in this case we say that the two $p$-colorings differ only by a permutation of the colors. This definition of $p$-coloring classes corresponds directly to the $\bmod p$ rank discussed in chapter 3 of [9].

**3. p-Colorability of Torus Knots.** Let $T_{m,n}$ represent the torus knot characterized by the number of times $m$ that it circles around the meridian of the torus and the number of times $n$ that it circles around the longitude of the torus. $T_{m,n}$ has one component if and only if $m$ and $n$ are relatively prime. It is well-known that every knot is the closure of some braid [9]. For example, the trefoil knot $T_{3,2}$ is the closure of the braid $(\sigma_1\sigma_2)^2$ shown in Figure 1, where $\sigma_i$ represents a crossing where string $i-1$ crosses over string $i$. In general, the torus knot $T_{m,n}$ can be realized as the closure of the braid word $(\sigma_1\sigma_2 \cdots \sigma_{m-2}\sigma_{m-1})^n$.

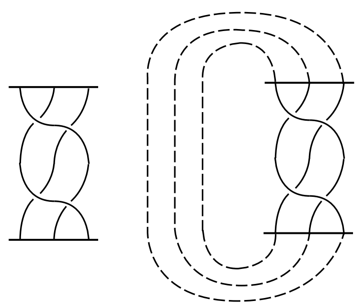

Figure 1: The braid $(\sigma_1\sigma_2)^2$ and its closure $T_{3,2}$.

Since $T_{m,n}$ is equivalent to $T_{n,m}$, the following theorem completely characterizes the $p$-colorability of torus knots. Note that if $T_{m,n}$ has one component, then $m$ and $n$ cannot both be even. Results similar to those in Theorem 1 were stated without proof by Asami and Satoh in [3]. We present an elementary proof here.

Theorem 1. Suppose $T_{m,n}$ is a torus knot and $p$ is prime.

i) If $m$ and $n$ are both odd, then $T_{m,n}$ is not $p$-colorable.
ii) If $m$ is odd and $n$ is even, then $T_{m,n}$ is $p$-colorable if and only if $p|m$.

Proof. If $p$ is prime, then a knot $K$ is $p$-colorable if and only if $p$ divides $\det(K)$ [9]. We will show that the determinant of $T_{m,n}$ is

$$\det(T_{m,n}) = \begin{cases} 1, & \text{if } m \text{ and } n \text{ are both odd;} \\ m, & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

By [10] we have $\det(K) = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of $K$. The Alexander polynomial for $T_{m,n}$ is given by [5]

$$\Delta_{T_{m,n}}(t) = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}.$$

Therefore, if $m$ and $n$ are both odd, we have $\Delta_{T_{m,n}}(-1) = \frac{(-2)(-2)}{(-2)(-2)} = 1$. If $m$ is odd and $n$ is even, then by L'Hôpital's rule we have $\Delta_{T_{m,n}}(-1) = \frac{(mn+1)+mn-1}{(m+n)-m+n} = m$.

**4. p-Coloring Classes of Torus Knots.** Our second theorem is a special case of a result found by Asami and Kuga in [2]. They prove that if a knot $T_{m,n}$ can be $p$-colored using a finite Alexander quandle, it has a total of $p^2$ trivial and nontrivial colorings. If $T_{m,n}$ cannot be colored by such a quandle, then it has only the $p$ trivial colorings. It is important to note that Asami and Kuga only consider the total number of all $p$-colorings without distinguishing between equivalent colorings, while we consider equivalence classes of $p$-colorings, or $p$-coloring classes. Also, note that the proof of Theorem 2 will show that every nontrivial $p$-coloring of the standard braid projection of $T_{m,n}$ must use all $p$ colors.

Theorem 2. If $p$ is prime and $T_{m,n}$ is $p$-colorable, then $T_{m,n}$ has only one nontrivial $p$-coloring class.

Proof. If $T_{m,n}$ is $p$-colorable, then by Theorem 1 we can assume, without loss of generality, that we have $m$ odd, $n$ even, and $p|m$. Given a $p$-coloring of $T_{m,n}$ in the standard $m$-stand braid projection, let its $j$th *color array* be the element of $(\mathbb{Z}_p)^m$ whose $i$th component is the color of the $i$th strand of the braid representation of $T_{m,n}$ after $j$ cycles. The map $\phi\colon (\mathbb{Z}_p)^m \to (\mathbb{Z}_p)^m$ defined by

$$\phi(c_0, c_1, \ldots, c_{m-1}) = (2c_0 - c_1, 2c_0 - c_2, \ldots, 2c_0 - c_{m-1}, c_0)$$

describes the transition from the $j$th to the $(j+1)$st color array of $\alpha$, according the rules of $p$-colorability, as seen in Figure 2. Note that a $p$-coloring of $T_{m,n}$ is entirely determined by its initial color array, and that to have a proper $p$-coloring it is necessary and sufficient that $\phi^n$ fixes this initial color array.

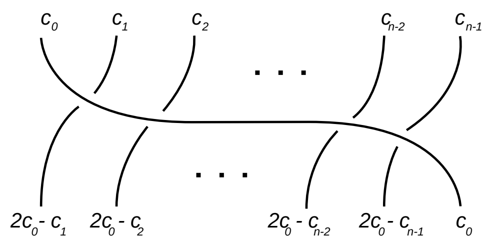


Figure 2: The action of $\phi$ on the $j$th color array of $T_{m,n}$.

Now since $p$ divides the number $m$ of braid strands in our projection of $T_{m,n}$, we can consider the 0th color array that consists of the colors $0, 1, \ldots, p-1$ listed in order $\frac{m}{p}$ times:

$$C_0 = (0, 1, \ldots, p-2, p-1,\ 0, 1, \ldots, p-2, p-1,\ \ldots, p-2, p-1).$$

Under the action of $\phi$, the $1^{\text{st}}$ color array is clearly

$$\phi(C_0) = (p-1, p-2, \ldots, 1, 0,\ p-1, p-2, \ldots, 1, 0,\ \ldots, 1, 0),$$

and the $2^{\text{nd}}$ color array is

$$\phi^2(C_0) = (0, 1, \ldots, p-2, p-1,\ 0, 1, \ldots, p-2, p-1,\ \ldots, p-2, p-1).$$

Since $\phi^2$ fixes $C_0$ and $n$ is even, we know that $\phi^n$ fixes $C_0$ and thus, the initial color array $C_0$ induces a nontrivial $p$-coloring.

Note that the $p$-coloring constructed above has the property that its initial color array $c_0, c_1, c_2, \ldots, c_{m-1}$ has *constant variance* of 1, since $c_{j+1} - c_j = 1$ for all $0 \leq j \leq m$ (with indices mod $m$). It can be shown by elementary, but tedious, methods that if $T_{m,n}$ is a one-component link, then any $p$-coloring of its standard $m$-strand braid projection will have constant variance (not necessarily equal to 1), and that all such $p$-colorings are equivalent to the $p$-coloring constructed above. Thankfully, the reviewer for this paper suggested a much more elegant method of proving that, up to equivalence, there can be no more than one nontrivial $p$-coloring of $T_{m,n}$, as follows.

Seeking a contradiction, suppose that there are two non-equivalent nontrivial $p$-colorings $c_1, c_2, \ldots, c_r,\ d_1, d_2, \ldots, d_r$ of a projection of $T_{m,n}$ with strands $s_1, s_2, \ldots, s_r$. We will show that these colorings induce what we can think of as a $\mathbb{Z}_p \oplus \mathbb{Z}_p$-coloring $(c_1, d_1), \ldots (c_r, d_r)$ of the strands of the projection in the sense that we have an onto homomorphism from the knot group $\theta$ from $\pi_1(S^3 - K)$ to the generalized Dihedral group $\mathcal{D} = \langle a, b \mid a^2 = 1, b \in \mathbb{Z}_p \oplus \mathbb{Z}_p, abab = 1\rangle$.

In the notation above, and writing $b = (b_1, b_2)$, define $\theta(x_i) = a(b_1^{c_i}, b_2^{d_i})$. We will show that $\theta$ is onto by showing that $a$, $(b_1, 1)$, and $(1, b_2)$ are in its image. By non-equivalence, there must exist some $i, j$ such that either $c_i = c = j$ but $d_i \neq d_j$, or $d_i = d_j$ but $c_i \neq c_j$. Without loss of generality we will assume the former. With this $i, j$, it is a simple exercise to show that $\phi(x_i x_j) = (1, b_2^{d_j - d_i})$. Since $d_j - d_i \neq 0$ and $p$ is prime, some power of this element is $(1, b_2)$. Now since $c_1, c_2, \ldots, c_r$ is a nontrivial $p$-coloring, there must exist some $k$ such that $c_i \neq c_k$, and for this $i, k$ we have $\phi(x_i x_k) = (b_1^{c_k - c_i}, b_2^{d_k - d_i})$. The product of this element with $(1, b_2)^{d_i - d_k}$ is $(b_1^{c_k - c_i}, 1)$, and again since $c_k - c_i \neq 0$ and $p$ is prime, some power of this is $(b_1, 1)$. We now see immediately that $a$ is in the image of $\theta$, since $\theta(x_i)(b_1^{-c_i}, b_2^{-d_i}) = a$.

The existence of this onto map $\theta$ provides a contradiction to there being two non-equivalent $p$-colorings, as follows. It is well-known [5] that $\pi_1(S^3 - T_{m,n}) = \langle x, y \mid x^m = y^n\rangle$, and that the center $Z$ of this group is generated by $x^m$. Since $\theta$ is onto and thus, carries centers into centers, $\theta(Z)$ is contained in the center of $\mathcal{D}$, which is trivial since $p$ is odd. Therefore, $Z \in \ker\theta$, and thus, the map $\theta$ factors through the group $\langle x, y \mid x^m = 1, y^m = 1\rangle$. This induces a map $\beta\colon \langle x, y \mid x^m = 1, y^m = 1\rangle \to \mathcal{D}$, which must be onto since $\theta$ is onto. But there can be no onto homomorphism from a free product of two cyclic groups to a group whose presentation requires at least three generators. Therefore, there can be only one nontrivial $p$-coloring class for $T_{m,n}$, namely the one we constructed above.

Notice that the proof of Theorem 2 shows that any $p$-coloring of the standard minimal projection of a torus knot must use all $p$ colors. In particular, this gives another proof that torus knots of the form $T_{p,2}$ for $p$ an odd prime satisfy the Harary-Kauffman Conjecture [7]; such torus knots are alternating with determinant $p$, and the least number of colors needed to nontrivially color a minimal projection of $T_{p,2}$ will be equal to the crossing number $p$.

The reviewer for this paper pointed out to the authors that another immediate consequence of our elementary result in Theorem 2 is that the first homology groups of certain $q$-fold branched cyclic covers of torus knot complements are cyclic. The requirement that this homology group be cyclic in the two-fold case has been suggested as a weaker hypothesis for the Harary-Kauffman Conjecture [1]. Given a torus knot $T_{m,n}$, let $\widehat{C}^q_{m,n}$ denote the $q$-fold branched cyclic cover of $S^3 - T_{m,n}$ [5].

Corollary 3. If $q = 2 + kmn$ for some nonnegative integer $k$, then the homology group $H_1(\widehat{C}^q_{m,n})$ is cyclic.

Proof. By [5], $T_{m,n}$ is $p$-colorable for some prime $p$ if and only if $p$ divides $|H_1(\widehat{C}^2_{m,n})|$. For each such prime $p$, Theorem 2 shows that there is only one nontrivial $p$-coloring class, which in turn guarantees that the 2-fold branched cyclic cover of the knot complement contains only one subgroup of order $p$ [11]. Since this is true for all primes $p$ that divide $|H_1(\widehat{C}^2_{m,n})|$, the result follows in the $q = 2$ (i.e. $k = 0$) case. The general result now follows from the fact that the $q$-fold coverings have period $mn$ [5], i.e. $\widehat{C}^q_{m,n} \cong \widehat{C}^{q+kmn}_{m,n}$ for any nonnegative integer $k$.

The authors of this paper would like to thank the reviewer, who made extremely detailed comments and in addition, suggested a more streamlined argument and a way to extend our original results (Corollary 3). This work was supported by NSF grant number NSF-DMS 0243845.

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Mathematics Subject Classification (2000): 57M27, 05C15

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