A TREE FOR COMPUTING THE
CAYLEY-DICKSON TWIST

John W. Bales

Abstract. A universal twist γ for all finite-dimensional Cayley-Dickson algebras is defined recursively and a tree diagram 'computer' is presented for determining the value of $\gamma(p,q)$ for any two non-negative integers p and q.

1. Introduction. The Cayley-Dickson algebras are a nested sequence $\{\mathbb{A}_k\}$ of algebras with $\mathbb{A}_k \subset \mathbb{A}_{k+1}$. $\mathbb{A}_0 = \mathbb{R}$ and for any $k \geq 0$, $\mathbb{A}_{k+1}$ consists of all ordered pairs of elements of $\mathbb{A}_k$ with a conjugate defined by [8]

$$(x,y)^* = (x^*, -y) \tag{1.1}$$

and multiplication by

$$(a,b)(c,d) = (ac - db^*, a^*d + cb). \tag{1.2}$$

For $k \geq 0$, the real number a is identified with the ordered pair $(a,0)$ in $\mathbb{A}_{k+1}$. Accordingly, scalar multiplication of an ordered pair (c,d) in $\mathbb{A}_{k+1}$ by a real number a is $(a,0)(c,d) = (ac, ad)$. The first few algebras in this sequence are $\mathbb{A}_0 = \mathbb{R}$ the reals, $\mathbb{A}_1 = \mathbb{C}$ the complex numbers, $\mathbb{A}_2 = \mathbb{H}$ the quaternions, $\mathbb{A}_3 = \mathbb{O}$ the octonions, and $\mathbb{A}_4 = \mathbb{S}$ the sedenions [2].

In the following development of the sequence of Cayley-Dickson algebras, every element of each algebra will be identified with an infinite sequence of real numbers terminating in a sequence of zeros. A real number r is identified with the sequence $r, 0, 0, 0, \ldots$. If each $x = x_0, x_1, x_2, \ldots$ and $y = y_0, y_1, y_2, \ldots$ is an element in one of the Cayley-Dickson algebras, then the ordered pair (x,y) is the sequence constructed by "shuffling" the two sequences.

$$(x,y) = x_0, y_0, x_1, y_1, x_2, y_2, \ldots . \tag{1.3}$$

For example, a complex number (a,b) is formed by shuffling the sequences $a, 0, 0, 0, \ldots$ and $b, 0, 0, 0, \ldots$ to obtain $(a,b) = a, b, 0, 0, 0, \ldots$. The quaternion $((a,b),(c,d))$ is formed by shuffling the sequences $a, b, 0, 0, 0, \ldots$ and $c, d, 0, 0, 0, \ldots$ to obtain the sequence $a, c, b, d, 0, 0, 0 \ldots$. This process may be repeated indefinitely to represent all Cayley-Dickson elements as finite sequences terminated by an infinite sequence of zeros. This construction of the elements of the algebras leads to a universal Cayley-Dickson algebra $\mathbb{A}$ which is simply the union of all the algebras with each $\mathbb{A}_k$ a proper subspace of $\mathbb{A}$. Furthermore, if $a, b \in \mathbb{A}$, then the ordered pair $(a,b) \in \mathbb{A}$. Since $\mathbb{A} \subseteq \ell^2$, the Hilbert space of square summable sequences, and since ℓ^2 is a completion of $\mathbb{A}$, it is natural to ask whether the product

defined on $\mathbb{A}$ can be extended to ℓ^2 and whether ℓ^2 is closed under that product. The answer to the first question is 'yes' and the answer to the second is unknown.

The sequences

$$i_0 = (1, 0) = 1, 0, 0, 0, \ldots$$
$$i_1 = (0, 1) = 0, 1, 0, 0, \ldots$$
$$i_2 = (i_1, 0) = 0, 0, 1, 0, \ldots$$
$$i_3 = (0, i_1) = 0, 0, 0, 1, \ldots$$
$$\vdots$$
$$i_{2^n - 1}$$

form the canonical basis for $\mathbb{A}_n$ and satisfy the identities

$$i_{2k} = (i_k, 0) \tag{1.4}$$
$$i_{2k+1} = (0, i_k) \tag{1.5}$$
$$i_p^* = \begin{cases} i_p, & \text{if } p = 0; \\ -i_p, & \text{if } p > 0. \end{cases} \tag{1.6}$$

One may establish immediately that $i_0 = (1, 0)$ is both the left and the right identity for $\mathbb{A}_n$. Furthermore, applying the Cayley-Dickson product (1.2) to all the unit basis vectors yields the following identities:

$$i_{2p} i_{2q} = (i_p, 0)(i_q, 0) = (i_p i_q, 0) \tag{1.7}$$
$$i_{2p} i_{2q+1} = (i_p, 0)(0, i_q) = (0, i_p^* i_q) \tag{1.8}$$
$$i_{2p+1} i_{2q} = (0, i_p)(i_q, 0) = (0, i_q i_p) \tag{1.9}$$
$$i_{2p+1} i_{2q+1} = (0, i_p)(0, i_q) = -(i_q i_p^*, 0). \tag{1.10}$$

For $p = q = 0$ this generates the multiplication table for complex numbers (Table 1).

	i_0	i_1
i_0	i_0	i_1
i_1	i_1	$-i_0$

Table 1. Multiplication Table for Complex Number Basis Vectors

For $0 \le p \le 1$ and $0 \le q \le 1$ one obtains, using binary notation, the multiplication table for quaternions (Table 2).

	i_{00}	i_{01}	i_{10}	i_{11}
i_{00}	i_{00}	i_{01}	i_{10}	i_{11}
i_{01}	i_{01}	$-i_{00}$	i_{11}	$-i_{10}$
i_{10}	i_{10}	$-i_{11}$	$-i_{00}$	i_{01}
i_{11}	i_{11}	i_{10}	$-i_{01}$	$-i_{00}$

Table 2. Multiplication Table for Quaternion Basis Vectors

For $0 \leq p \leq 2$ and $0 \leq q \leq 2$ one obtains the multiplication table for octonions (Table 3).

	i_{000}	i_{001}	i_{010}	i_{011}	i_{100}	i_{101}	i_{110}	i_{111}
i_{000}	i_{000}	i_{001}	i_{010}	i_{011}	i_{100}	i_{101}	i_{110}	i_{111}
i_{001}	i_{001}	$-i_{000}$	i_{011}	$-i_{010}$	i_{101}	$-i_{100}$	i_{111}	$-i_{110}$
i_{010}	i_{010}	$-i_{011}$	$-i_{000}$	i_{001}	i_{110}	$-i_{111}$	$-i_{100}$	i_{101}
i_{011}	i_{011}	i_{010}	$-i_{001}$	$-i_{000}$	$-i_{111}$	$-i_{110}$	i_{101}	i_{100}
i_{100}	i_{100}	$-i_{101}$	$-i_{110}$	i_{111}	$-i_{000}$	i_{001}	i_{010}	$-i_{011}$
i_{101}	i_{101}	i_{100}	i_{111}	i_{110}	$-i_{001}$	$-i_{000}$	$-i_{011}$	$-i_{010}$
i_{110}	i_{110}	$-i_{111}$	i_{100}	$-i_{101}$	$-i_{010}$	i_{011}	$-i_{000}$	i_{001}
i_{111}	i_{111}	i_{110}	$-i_{101}$	$-i_{100}$	i_{011}	i_{010}	$-i_{001}$	$-i_{000}$

Table 3. Multiplication Table for Octonion Basis Vectors

Representing the basis vectors with binary subscripts reveals that the product of i_p and i_q is a multiple of the basis vector subscripted by the sum of p and q in $\mathbb{Z}_2^n$. This is equivalent to the bit-wise 'exclusive or' of the binary numbers p and q. The multiple is either $+1$ or -1. The basis vectors are indexed by elements of the group $\mathbb{Z}_2^n$ with a 'twist' γ [6]. That is, there is a function $\gamma \colon \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \{-1, 1\}$ such that for all $p, q \in \mathbb{Z}_2^n$,

$$i_p i_q = \gamma(p, q) i_{pq}, \tag{1.11}$$

where pq represents the sum of p and q in the group $\mathbb{Z}_2^n$. The elements of the group $\mathbb{Z}_2^n$ may be regarded as integers ranging from 0 to $2^n - 1$ with group operation the 'exclusive or' of the binary representations. This operation is equivalent to addition in $\mathbb{Z}_2^n$.

If G is a group and $\mathbb{K}$ is a ring, then one may construct a 'group algebra' $\mathbb{A}(\mathbb{K})$ consisting of all elements $\sum_{r \in G} k_r i_r$, where $k_r \in \mathbb{K}$ and i_r is a basis vector in $\mathbb{A}(\mathbb{K})$. The product of basis vectors i_p and i_q is $i_p i_q = i_{pq}$. This scheme may be modified by adding a 'twist.' A twist is a function α from $G \times G$ to $\{-1, 1\}$. For the twisted group algebra, the product of the basis vectors is $i_p i_q = \alpha(p, q) i_{pq}$.

2. The Recursive Definition of the Universal Cayley-Dickson Twist γ.

<u>Theorem 2.1</u>. There is a twist $\gamma(p, q)$ mapping $\cup \mathbb{Z}_2^k \times \cup \mathbb{Z}_2^k$ onto $\{-1, 1\}$ such that if $p, q \in \cup \mathbb{Z}_2^k$, then $i_p i_q = \gamma(p, q) i_{pq}$.

Proof. Assume $0 \leq p < 2^n$ and $0 \leq q < 2^n$ and proceed by induction on n using (1.4)–(1.10).

If $n = 0$, then $p = q = 0$ and $i_p i_q = i_0 i_0 = i_0 = \gamma(p, q) i_{pq}$ provided $\gamma(0, 0) = 1$.

Suppose the principle is true for $n = k$. Let $0 \leq p < 2^{k+1}$ and $0 \leq q < 2^{k+1}$. Then there are numbers r and s such that $0 \leq r < 2^k$ and $0 \leq s < 2^k$ and such that one of the following is true:

- $p = 2r, \ q = 2s$
- $p = 2r, \ q = 2s + 1$
- $p = 2r + 1, \ q = 2s$
- $p = 2r + 1, \ q = 2s + 1$

(1) Assume $p = 2r, \ q = 2s$. Then

$$
\begin{aligned}
i_p i_q = i_{2r} i_{2s} &= (i_r i_s, 0) \\
&= (\gamma(r, s) i_{rs}, 0) = \gamma(r, s)(i_{rs}, 0) \\
&= \gamma(r, s) i_{2rs} = \gamma(2r, 2s) i_{(2r)(2s)} \\
&= \gamma(p, q) i_{pq},
\end{aligned}
$$

provided $\gamma(2r, 2s) = \gamma(r, s)$.

(2) Assume $p = 2r, \ q = 2s + 1$. Then $i_p i_q = i_{2r} i_{2s+1} = (0, i_r^* i_s)$.
If $r \neq 0$, then

$$
\begin{aligned}
i_p i_q &= -(0, i_r i_s) = -(0, \gamma(r, s) i_{rs}) \\
&= -\gamma(r, s) i_{2rs+1} = \gamma(2r, 2s + 1) i_{(2r)(2s+1)} \\
&= \gamma(p, q) i_{pq},
\end{aligned}
$$

provided $\gamma(2r, 2s + 1) = -\gamma(r, s)$ when $r \neq 0$.
If $r = 0$, then

$$
\begin{aligned}
i_p i_q = i_0 i_{2s+1} &= (0, i_0 i_s) \\
&= (0, \gamma(0, s) i_s) = \gamma(0, s) i_{2s+1} \\
&= \gamma(0, 2s + 1) i_{pq} = \gamma(p, q) i_{pq},
\end{aligned}
$$

provided $\gamma(0, 2s + 1) = \gamma(0, s)$.

(3) Assume $p = 2r + 1, \ q = 2s$. Then

$$
\begin{aligned}
i_p i_q = i_{2r+1} i_{2s} &= (0, i_s i_r) \\
&= \gamma(s, r)(0, i_{sr}) = \gamma(s, r) i_{2sr+1} \\
&= \gamma(2r + 1, 2s) i_{(2r+1)(2s)} = \gamma(p, q) i_{pq},
\end{aligned}
$$

provided $\gamma(2r + 1, 2s) = \gamma(s, r)$.

(4) Assume $p = 2r + 1, \ q = 2s + 1$. Then $i_p i_q = i_{2r+1} i_{2s+1} = -(i_s i_r^*, 0)$.

If $r \neq 0$, then

$$
\begin{aligned}
i_p i_q = (i_s i_r, 0) &= \gamma(s, r)(i_{sr}, 0) \\
&= \gamma(s, r) i_{2sr} = \gamma(2r+1, 2s+1) i_{(2r+1)(2s+1)} \\
&= \gamma(p, q) i_{pq},
\end{aligned}
$$

provided $\gamma(2r+1, 2s+1) = \gamma(s, r)$ when $r \neq 0$.

If $r = 0$, then

$$
\begin{aligned}
i_p i_q = i_1 i_{2s+1} &= -(i_s i_0^*, 0) \\
&= -(i_s i_0, 0) = -\gamma(s, 0)(i_s, 0) \\
&= -\gamma(s, 0) i_{2s} = \gamma(1, 2s+1) i_{1(2s+1)} \\
&= \gamma(p, q) i_{pq},
\end{aligned}
$$

provided $\gamma(1, 2s+1) = -\gamma(s, 0)$.

Thus, the principle is true for $n = k+1$ provided the twist is defined as required in these four cases.

The properties of the twist γ may be summarized as follows:

$$\gamma(0,0) = \gamma(p, 0) = \gamma(0, q) = 1 \tag{2.1}$$

$$\gamma(2p, 2q) = \gamma(p, q) \tag{2.2}$$

$$\gamma(2p+1, 2q) = \gamma(q, p) \tag{2.3}$$

$$\gamma(2p, 2q+1) = \begin{cases} -\gamma(p, q) & \text{if } p \neq 0; \\ 1 & \text{otherwise} \end{cases} \tag{2.4}$$

$$\gamma(2p+1, 2q+1) = \begin{cases} \gamma(q, p) & \text{if } p \neq 0; \\ -1 & \text{otherwise.} \end{cases} \tag{2.5}$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1
\end{pmatrix}
$$

Table 4. Sedenion Twist Table

Table 4 is the twist table for $\mathbb{A}_4$, the sedenions–the fourth algebra in the sequence of algebras formed by the Cayley-Dickson process. The sedenions consist of all ordered pairs of octonions. The rows and columns of Table 4 are numbered 0 through 15. The entry in row p column q is $\gamma(p, q)$. The sedenion twist table subsumes the twist tables for the octonions, quaternions, complex numbers and reals in the following sense. The submatrix formed by rows 0–7 and columns 0–7 is the twist table for the octonions, rows 0–3 columns 0–3 the twist table for the quaternions, etc. The table reveals a uniform structure which is common to the twist tables of all higher order Cayley-Dickson algebras.

In general, the twist table for $\mathbb{Z}_2^n$ may be partitioned into 2×2 matrices, where each 2×2 matrix is one of three matrices or the negatives of two of those three matrices.

To see why this is the case, reform the recursive definition 2.1–2.5 of the Cayley-Dickson twist on $\cup \mathbb{Z}_2^k$ as follows: For $p, q \in \cup \mathbb{Z}_2^k$ and $r, s \in \{0, 1\}$,

$$\gamma(0, 0) = 1 \tag{2.6}$$
$$\gamma(2p + r, 2q + s) = \gamma(p, q) E_{pq}(r, s), \tag{2.7}$$

where

$$E_{pq} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ if } p = 0 \tag{2.8}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ if } 0 \neq p = q \text{ or } p \neq q = 0 \tag{2.9}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \text{ if } 0 \neq p \neq q \neq 0. \tag{2.10}$$

Define $\gamma_0 = (1)$. Then for each non-negative integer n, γ_{n+1} is a partitioned matrix defined by

$$\gamma_{n+1} = (\gamma_n(p, q) E_{pq}). \tag{2.11}$$

This arrangement is interesting enough, but the structure of the table can also be analyzed in a different way using the same three matrices.

<u>Theorem 2.2</u>. For $n > 0$, the Cayley-Dickson twist table γ_n can be partitioned into 2×2 blocks of matrices A, B, C, $-B$, or $-C$, defined as follows:

$$A_0 = A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

<u>Proof</u>. To construct γ_{n+1} from γ_n, replace each entry $\gamma(p, q)$ in γ_n by the 2×2 block

$$E_{pq} = \begin{pmatrix} \gamma(2p, 2q) & \gamma(2p, 2q + 1) \\ \gamma(2p + 1, 2q) & \gamma(2p + 1, 2q + 1) \end{pmatrix}. \tag{2.12}$$

When $p = 0$,

$$E_{0q} = \begin{pmatrix} \gamma(0, 2q) & \gamma(0, 2q+1) \\ \gamma(1, 2q) & \gamma(1, 2q+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.13}$$

When $p \neq 0$ we have

$$E_{pq} = \begin{pmatrix} \gamma(p, q) & -\gamma(p, q) \\ \gamma(q, p) & \gamma(q, p) \end{pmatrix}. \tag{2.14}$$

There are only five possible values for the 2×2 matrix E_{pq} in (2.14). These may be found by considering the following cases:

(1) $0 = p = q$
(2) $0 = p \neq q$
(3) $p \neq q = 0$
(4) $p = q \neq 0$
(5) $0 \neq p \neq q \neq 0$.

For the first two cases, $E_{pq} = E_{0q} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = A$.

For the third case, $E_{pq} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = B$.

For the fourth case, $E_{pq} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} = -B$.

For the fifth case, $E_{pq} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = C$ when $\gamma(p, q) = 1$ and $E_{pq} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -C$ when $\gamma(p, q) = -1$.

The first few tables are displayed in Tables 5 through 8.

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = A_0$$

Table 5. Complex Twist Table

$$\gamma_2 = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{array} \right) = \left(\begin{array}{c|c} A_0 & A \\ \hline B & -B \end{array} \right)$$

Table 6. Quaternion Twist Table

$$
\gamma_3 = \left(\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
\hline
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
\hline
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\hline
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1
\end{array}\right)
$$

$$
= \left(\begin{array}{c|c|c|c}
A_0 & A & A & A \\
\hline
B & -B & C & -C \\
\hline
B & -C & -B & C \\
\hline
B & C & -C & -B
\end{array}\right)
$$

Table 7. Octonion Twist Table

$$
\gamma_4 = \left(\begin{array}{cc|cc|cc|cc}
A_0 & A & A & A & A & A & A & A \\
B & -B & C & -C & C & -C & C & -C \\
\hline
B & -C & -B & C & C & -C & -C & C \\
B & C & -C & -B & -C & -C & C & C \\
\hline
B & -C & -C & C & -B & C & C & -C \\
B & C & C & C & -C & -B & -C & -C \\
\hline
B & -C & C & -C & -C & C & -B & C \\
B & C & -C & -C & C & C & -C & -B
\end{array}\right)
$$

Table 8. Sedenion Twist Table

For $n > 0$, γ_n can be partitioned into 2×2 matrices, or blocks consisting of only A, B, $-B$, C, or $-C$. The first row of the partitioned γ_n will consist entirely of A blocks. The first such A block in row 1 will be denoted A_0. The first column of the partitioned γ_n, with the exception of A_0, will consist of B blocks. All blocks occurring along the diagonal of the partitioned γ_n, with the exception of A_0 will be $-B$. All other blocks of the partitioned γ_n will consist of either C or $-C$. Notice that the signs of the entries in γ_n are the same as the signs of the corresponding blocks in γ_{n+1}.

The function T maps each of the six 2×2 blocks A_0, A, B, $-B$, C, and $-C$ into a 4×4 block according to the following rules.

$$T(A_0) = \begin{pmatrix} A_0 & A \\ B & -B \end{pmatrix} \tag{2.15}$$

$$T(A) = \begin{pmatrix} A & A \\ C & -C \end{pmatrix} \tag{2.16}$$

$$T(B) = \begin{pmatrix} B & -C \\ B & C \end{pmatrix} \tag{2.17}$$

$$T(-B) = \begin{pmatrix} -B & C \\ -C & -B \end{pmatrix} \tag{2.18}$$

$$T(C) = \begin{pmatrix} C & -C \\ -C & -C \end{pmatrix} \tag{2.19}$$

$$T(-C) = \begin{pmatrix} -C & C \\ C & C \end{pmatrix}. \tag{2.20}$$

For a given twist table γ_n, let $T(\gamma_n)$ denote the matrix which results by replacing each occurrence of A_0, A, B, $-B$, C, and $-C$ in γ_n with the 2×2 blocks $T(A_0)$, $T(A)$, $T(B)$, $T(-B)$, $T(C)$, and $T(-C)$, respectively. Then the Cayley-Dickson twist tables can be defined recursively as follows:

(1) $\gamma_1 = A_0$
(2) $\gamma_{n+1} = T(\gamma_n)$.

This process can be summarized by the tree in Figure 9 which for clarity is broken into its components. The root of the tree is A_0.
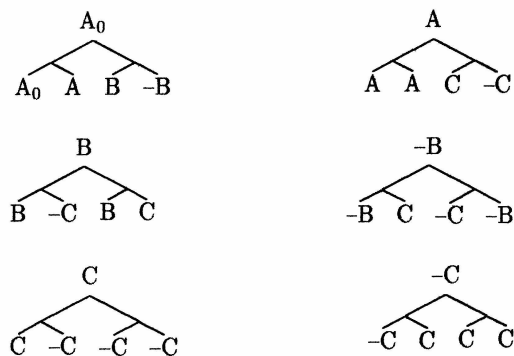

Table 9. Cayley-Dickson Twist Tree

The Cayley-Dickson twist for any pair of non-negative integers may be found using this tree.

Example. Use the Cayley-Dickson tree to find $\gamma(9, 11)$.

(1) First, rewrite 9 and 11 in binary form 1001 and 1011.
(2) Next, "shuffle" the two numbers to obtain 11000111. It will be easier to use this bit string if it is separated by commas into pairs 11, 00, 01, 11. Each pair, beginning at the *left* of the string is a navigation instruction for the Cayley-Dickson tree. A 0 is an instruction to move down the left branch and a 1 is an instruction to move down the right branch.
(3) Beginning at the root A_0 and applying the first instruction 11 on the left yields $A_0(11) = -B$.
(4) Beginning at $-B$ and applying the next instruction 00 yields $-B(00) = -B$. (Notice that 00 always leaves the state unchanged.)
(5) Beginning at $-B$ and applying the next instruction 01 yields $-B(01) = C$.
(6) Finally, beginning at C and applying the final instruction 11 yields $C(11) = -C$. Since the 'coefficient' of C is -1, $\gamma(9, 11) = -1$.
(7) This particular traversal of the tree may be summarized as follows:

$$\gamma(9, 11) = \gamma(1001, 1011) \rightarrow 11, 00, 01, 11 \rightarrow -B, -B, C, -C \rightarrow (-1).$$

3. Conclusion. The Cayley-Dickson twist for any pair of non-negative integers may be found using the recursive definition (2.1) or by traversing the Cayley-Dickson tree.

The fact that a universal twist γ on $\cup \mathbb{Z}_2^k$ exists establishes that the set $\mathbb{A} = \cup \mathbb{A}_k$ of all finite sequences of real numbers is a Cayley-Dickson algebra. The twist γ on the group $\cup \mathbb{Z}_2^k$ is a *proper* [3] twist, meaning that it satisfies the properties

$$\gamma(p, q)\gamma\left(q, q^{-1}\right) = \gamma\left(pq, q^{-1}\right) \tag{3.1}$$

$$\gamma\left(p^{-1}, p\right)\gamma(p, q) = \gamma\left(p^{-1}, pq\right). \tag{3.2}$$

All properly twisted group algebras, including the Cayley-Dickson and Clifford algebras, satisfy the *adjoint properties* [10,3]. The adjoint properties state that for elements x, y, and z of the algebra,

$$\langle xy, z \rangle = \langle y, x^* z \rangle \tag{3.3}$$

$$\langle x, yz \rangle = \langle xz^*, y \rangle \tag{3.4}$$

from which it follows that the Cayley-Dickson product of finite sequences x and y has [3] the Fourier expansion

$$xy = \sum_r \langle xy, i_r \rangle i_r = \sum_r \langle x, i_r y^* \rangle i_r = \sum_r \langle y, x^* i_r \rangle i_r. \tag{3.5}$$

If the product in (3.5) is applied to sequences x and y in ℓ^2 the resulting product xy is a well-defined number sequence, but it is not obviously square

summable. So it remains to be seen whether ℓ^2 is closed under the Cayley-Dickson product. If it were, it would be a universal topologically complete Cayley-Dickson algebra.

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John W. Bales
Department of Mathematics
Tuskegee University
Tuskegee, AL 36088, USA
email: jbales@tuskegee.edu