A NEW KIND OF CLUSTER SETS AND THEIR APPLICATIONS

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Abstract. In this article, we introduce s-cluster sets of functions and multifunctions and investigate their various properties. Such investigations provide a new technique for studying s-closed spaces [7] and weakly Hausdorff spaces [9].

1. Introduction

The notion of cluster set has been extensively studied in real and analytic function theory. The book of Collingwood and Lohwater [1], in this regard, is quite remarkable. In [12], Weston initiated the investigation of cluster sets for arbitrary functions between topological spaces and subsequently, cluster sets for functions and multifunctions in general topology have been investigated by, Hamlett [3,4], Joseph [5] and Mukherjee and Debray [8].

In this paper, we introduce a new kind of cluster sets under the terminology s-cluster set which provides a new technique for the study of s-closed spaces. An explicit expression of s-cluster set of a function in terms of filters and grills have been given. Several sufficient conditions for having degenerate s-cluster sets and for characterizing weakly Hausdorff spaces in terms of degeneracy of s-cluster sets of a suitable function are given. Finally, as an application, we have been able to achieve our goal for giving some new characterizations of s-closed spaces.

Throughout this paper, spaces X and Y mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of a space X, clA and $intA$ represent the closure of A and the interior of A, respectively. A subset A of a space X is said to be regular open (resp. regular closed) if $A = int\ clA$ (resp. $A = cl\ intA$) and A is called semi-open [6] if for some open set U, $U \subseteq A \subseteq clU$. The complement of a semi-open set is called semi-closed set. The set which is both semi-open as well as semi-closed is called the semi-regular set [7]. The set of all open sets (resp. semi-open sets, semi-regular sets), each containing a given point

x of the space (X,τ) or simply of X is denoted by $\tau(x)$ (resp. $SO(x)$, $SR(x)$). The set of all semi-open sets, each containing a given subset A of X is denoted by $SO(A)$. Crossley and Hildebrand [2] has defined the semi-closure of a subset A, denoted by $sclA$, as the intersection of all semi-closed sets containing A. For a subset A of a space X, the θ-closure [11] (resp. δ-closure [11], semi-θ-closure [7]) of A, denoted by $\theta - clA$ (resp. $\delta - clA$, $scl_\theta A$), is the set of all points x of X such that $clU \cap A \neq \emptyset$ for each $U \in \tau(x)$ (resp. $int\ clU \cap A \neq \emptyset$ for each $U \in \tau(x)$, $sclU \cap A \neq \emptyset$ for each $U \in SO(x)$). While a set A is called θ-closed [11] (resp. δ-closed [11], semi-θ-closed [7]) if $A = \theta - clA$ (resp. $A = \delta - clA$, $A = scl_\theta A$).

A filter base $\mathcal{F}$ on a space X is said to SR-adhere [7] at x of X if $F \cap V \neq \emptyset$, for each $F \in \mathcal{F}$ and each $V \in SR(x)$. A filter base $\mathcal{F}$ on X is said to be SR-converge [7](resp. δ-converge) at x of X if for each $V \in SR(x)$ (resp. $V \in \tau(x)$) there is an $F \in \mathcal{F}$ such that $F \subset V$ (resp. $F \subset int\ clV$). Throne [10] has defined a non-empty family $\mathcal{G}$ of non-empty subsets of X to be a grill if (i) $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$ and (ii) $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

A grill $\mathcal{G}$ on X is said to SR-converge to a point x of (X,τ) if for each $V \in SR(x)$ there is some $G \in \mathcal{G}$ with $G \subseteq V$. A space X is called weakly Hausdorff space [9] if each point x of X is the intersection of all regular closed sets containing x. A subset A of a space X is said to be s-closed relative to X [7] if for every cover $\mathcal{U}$ of A by semi-open sets of X, there is a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subset \bigcup\{sclU : U \in \mathcal{U}_0\}$. If, in particular, $A = X$ then X is called an s-closed space [7]. As usual, by a multifunction $F : X \to Y$, we mean a function $F : X \to 2^Y - \{\emptyset\}$, where 2^Y is the power set of Y.

2. s-cluster Set of Function

Definition 2.1. *Let f be a map from a space (X,τ) into a space (Y,τ'). Then the s-cluster set of f at $x \in X$, denoted by $s_\tau(f,x)$ or $s(f,x)$ (when no topology is mentioned on X), is defined to be the set $\cap\{\delta - clf(sclV) : V \in SO(x)\}$.*

Theorem 2.2. *Let f be a function from the space X into the space Y. Then the following statements are equivalent:*

(i) $z \in s(f,x)$
(ii) *There is a grill $\mathcal{G}$ on X such that $\mathcal{G}$ SR-converges to x and $z \in \cap\{\delta - clf(G) : G \in \mathcal{G}\}$.*
(iii) *The filter base $\mathcal{F} = \{f^{-1}(scl(U) : U \in \tau(z)\}$ SR-adheres to x.*
(iv) *There is a filter $\mathcal{F}$ on X such that $\mathcal{F}$ SR-converges to x and $f(\mathcal{F})$ δ-converges to z.*

Proof. (i) $\Rightarrow$ (iii). Let $z \in s(f,x)$. Then for each $U \in \tau(z)$ and each $V \in SO(x)$, $int\ cl(U) \cap f(sclV) \neq \emptyset$ and hence, $f^{-1}(sclU) \cap sclV \neq \emptyset$ (as for the open set U, $int\ clU = sclU$ [7]). Therefore, the filter base $\mathcal{F} = \{f^{-1}(sclU) : U \in \tau(z)\}$ SR-adheres at x.

(iii) $\Rightarrow$ (ii). The family $\mathcal{G} = \{G \subseteq X : G \cap F \neq \emptyset$, for each $F \in \mathcal{F}_0\}$ is obviously a grill on X, where $\mathcal{F}_0$ is the filter generated by the filter base $\mathcal{F} = \{f^{-1}(sclU) : U \in \tau(z)\}$. Since $\mathcal{F}$ SR-adheres at x (by hypothesis (iii)), $sclV \cap f^{-1}(sclU) \neq \emptyset$ for each $V \in SO(x)$ and $U \in \tau(z)$. Clearly, $sclV \cap F \neq \emptyset$, for each $F \in \mathcal{F}$ and each $V \in SO(x)$ and hence, $sclV \in \mathcal{G}$ for each $V \in SO(x)$. Therefore, $\mathcal{G}$ SR-converges to x. So, $f(G) \cap sclW \neq \emptyset$ for each $W \in \tau(z)$ and for each $G \in \mathcal{G}$. Thus, $z \in \delta - clf(G)$ for all $G \in \mathcal{G}$. Therefore, $z \in \cap\{\delta - clf(G) : G \in \mathcal{G}\}$.

(ii) $\Rightarrow$ (i). Suppose there is a grill $\mathcal{G}$ on X SR-converging to x and $z \in \{\delta - clf(G) : G \in \mathcal{G}\}$. Then $sclU \in \mathcal{G}$ for all $U \in SO(x)$ and $z \in \delta - clf(G)$ for each $G \in \mathcal{G}$. Therefore, $z \in \delta - clf(sclU)$ for all $U \in SO(x)$ and hence, $z \in s(f,x)$.

(iv) $\Rightarrow$ (iii). Let there be a filter, say $\mathcal{F}$ on X such that $\mathcal{F}$ SR-converges to x and $f(\mathcal{F})$ δ-converges to z and also let $U \in \tau(z)$. Then there is an $F \in \mathcal{F}$ such that $f(F) \subset int\ clU = sclU$ and hence, $F \subset f^{-1}(sclU)$. So, $f^{-1}(sclU) \in \mathcal{F}$ and hence, $\mathcal{F}$ is finer than the filter base $\mathcal{F}_0 = \{f^{-1}(sclU) : U \in \tau(z)\}$. Since $\mathcal{F}$ SR-converges to x, $\mathcal{F}_0 = \{f^{-1}(sclU) : U \in \tau(z)\}$ SR-adheres at x.

(iii) $\Rightarrow$ (iv). Let the filter base $\mathcal{F}_0 = \{f^{-1}(sclU) : U \in \tau(z)\}$ SR-adheres at x. Consider the filter $\mathcal{F}$ generated by the filter base $\mathcal{F}_0^\star = \{F_0 \cap sclV : V \in SO(x)$ and $F_0 \in \mathcal{F}_0\}$. Clearly, $\mathcal{F}$ SR-converges at x. Let $U \in \tau(z)$ and $G \in f(\mathcal{F})$. Then there exist an $F \in \mathcal{F}$ such that $f(F) \subset G$ and hence, $f(F_0 \cap sclV) \subset f(F) \subset G$ where $F_0 = f^{-1}(sclU)$ for some $U \in \tau(z)$. Now $f(F_0 \cap sclV) \subset f(F_0) \cap f(sclV) \subset sclU$ and hence, $sclU \in f(\mathcal{F})$. Therefore, $f(\mathcal{F})$ δ-converges to z as $sclU = int\ clU$ [7]. □

Theorem 2.3. *For any δ-closed map f from space X to space Y, if $f^{-1}(y)$ is θ-closed in X for each $y \in Y$, then the s-cluster set $s(f,x)$ is degenerate for each $x \in X$.*

Proof. Clearly, $s(f,x) = \cap\{\delta - clf(sclV) : V \in SO(x)\} \subseteq \cap\{\delta - clf(\delta - clV) : V \in SO(x)\}$. As f is δ-closed, $s(f,x) \subseteq \cap\{f(\delta - clV) : V \in SO(x)\}$. Since $f^{-1}(y)$ is θ-closed, there is an open set $U \in \tau(x)$ such that $clU \cap f^{-1}(y) = \emptyset$ for each $x \notin f^{-1}(y)$. This implies that $y \notin f(clU) = f(\delta - clU)$. So, $y \notin s(f,x)$ for each $y \neq f(x)$. Therefore, $s(f,x) = \{f(x)\}$. □

Theorem 2.4. *Let $f : X \to Y$ be a δ-closed injective map, where X is Hausdorff and Y is any space. Then $s(f, x)$ is degenerate for every $x \in X$.*

Proof. Since f is a δ-closed map, $s(f, x) = \cap\{\delta - cl f(sclV) : V \in SO(x)\} \subseteq \cap\{\delta - cl f(\delta - clV) : V \in SO(x)\} = \cap\{f(\delta - clV) : V \in SO(x)\}$. When $z(\neq x)$ is any point in X, then as X is Hausdorff, there exist disjoint $V \in \tau(x)$ and $U \in \tau(z)$ such that $V \cap int\ clU = \emptyset$. Hence, $z \notin \delta - clV$ and hence, $f(z) \notin f(\delta - clV)$. Therefore, $f(z) \notin s(f, x)$. Since f is injective, $s(f, x) = \{f(x)\}$. □

Definition 2.5. *A function $f : X \to Y$ (X, Y are topological spaces) is called strongly irresolute at $x \in X$ if for each $V \in SO(f(x))$ there is an $U \in SO(x)$ such that $f(sclU) \subset V$. f is called strongly irresolute on X if it is strongly irresolute at each point x of X.*

Theorem 2.6. *A space Y is weakly Hausdorff if and only if for any space X and any surjective strongly irresolute function $f : X \to Y$, $s(f, x)$ is degenerate for each $x \in X$.*

Proof. Suppose for a space X, $f : X \to Y$ is strongly irresolute. So, for each $x \in X$ and any $U \in SO(f(x))$, there is a $V \in SO(x)$ such that $f(sclV) \subset U$. Now, $s(f, x) = \cap\{\delta - cl f(sclW) : W \in SO(x)\} \subseteq \cap\{\delta - clU : U \in SO(f(x))\}$. Since Y is weakly Hausdorff, there exists a regular closed set V' containing $f(x)$ such that $z \notin V'$ for each $z \neq f(x)$. Now, $z \in X - V' = int\ cl(X - V')$. Since f is strongly irresolute, there exists a $U \in SO(x)$ such that $f(sclU) \subset V'$. As $int\ cl(X - V') \cap V' = \emptyset$, $z \notin \delta - cl f(sclU)$ and hence, $z \notin s(f, x)$. Therefore, $s(f, x) = \{f(x)\}$ for each $x \in X$.

Conversely, let $y_i = f(x_i)$, $i = 1, 2$ be two distinct points in Y. By hypothesis, $s(f, x_2) = \{f(x_2)\} = \{y_2\}$ for any surjective strongly irresolute function $f : X \to Y$. Then there exist $U \in \tau(y_1)$ and $V \in SO(x_2)$ such that $int\ clU \cap f(sclV) = \emptyset$. Hence, $f(sclV) \subset Y - int\ clU = W$ (a regular closed set). So, $y_2 = f(x_2) \in W$ but $y_1 \notin W$ (as $y_1 \in U$). Thus, Y is weakly Hausdorff. □

Definition 2.7. *Let X be a set. A collection I of subsets of X is called a σ-ideal of subsets of X if*

(i) *$P \in I$ and $Q \subseteq P$ implies $Q \in I$.*
(ii) *$P_n \in I$, $n = 1, 2, \ldots$ implies $\bigcup_{n=1}^{\infty} P_n \in I$*
(iii) *$X \notin I$.*

Definition 2.8. *For a σ-ideal I on a set X, the two topologies τ_1 and τ_2 on X are said to be equivalent modulo I, denoted by $\tau_1 \equiv \tau_2 \pmod{I}$, if for any subset A of X, the difference set $scl_\theta^{\tau_1} A \sim scl_\theta^{\tau_2} A \in I$, where $scl_\theta^{\tau_i} A$ is the semi-θ-closure of A in the space (X, τ_i) for $i = 1, 2$.*

Theorem 2.9. *Let $\tau_1 \equiv \tau_2$ (mod I) where τ_1 and τ_2 are two topologies on X and I is a σ-ideal of subsets of X. If $f : X \to Y$ is any mapping where Y is second countable, then $s_{\tau_1}(f, x) = s_{\tau_2}(f, x)$ for each $x \in X - P$ for some $P \in I$.*

Proof. Let $\{x \in X : s_{\tau_1}(f, x) \neq s_{\tau_2}(f, x)\} = M \cup N$ where $M = \{x \in X : s_{\tau_1}(f, x) \nsubseteq s_{\tau_2}(f, x)\}$ and $N = \{x \in X : s_{\tau_2}(f, x) \nsubseteq s_{\tau_1}(f, x)\}$. As Y is second countable, there is a countable base, say, $\mathcal{B} = \{B_1, B_2, \ldots\}$. Let $M_n = \{x \in X : x \in scl_\theta^{\tau_1} f^{-1}(sclB_n)$ and $x \notin scl_\theta^{\tau_2} f^{-1}(sclB_n)\}$, for $n = 1, 2, \ldots$. Clearly, all $M_n \in I$ and hence, $\bigcup_{n\geq 1} M_n \in I$. If $x \in M$ then there exists a $z \in s_{\tau_1}(f, x)$ but $z \notin s_{\tau_2}(f, x)$. Now, Theorem 2.2 implies that the filter base $\mathcal{F} = \{f^{-1}(sclU) : U \in \tau(z)\}$ SR-adheres at x in (X, τ_1) but not in (X, τ_2). Hence, there is an open set $W \in \tau(z)$ such that $x \notin scl_\theta^{\tau_2} f^{-1}(sclW)$. As $\mathcal{B}$ is a base for Y, there exists some B_n such that $z \in B_n \subset W$ and hence, $x \notin scl_\theta^{\tau_2} f^{-1}(sclB_n)$. But we always have $x \in scl_\theta^{\tau_1} f^{-1}(sclB_n)$. Therefore, $x \in M_n$. Thus $M \subseteq \bigcup_{n\geq 1} M_n$ and hence, $M \in I$. Similarly, $N \in I$. Therefore, $M \cup N \in I$. Hence, the proof is complete. □

3. s-cluster Set for Multifunctions and Characterizations of s-closed Spaces

Definition 3.1. *Let $F : X \to Y$ be a multifunction from a space X into a space Y. Then the s-cluster set $s(F, x)$ of F at $x \in X$ is defined as the set $\cap\{\delta - clF(sclV) : V \in SO(x)\}$.*

Theorem 3.2. *Let $F : X \to Y$ be a multifunction from a space X into a space Y. If F has a δ-closed graph, then $s(F, x) = F(x)$, for each $x \in X$.*

Proof. Let $z \in s(F, x)$. Then for each $U \in \tau(z)$ and each $V \in SO(x)$, $int\ clU \cap F(sclV) \neq \emptyset$, i.e., $F^-(int\ clU) \cap sclV \neq \emptyset$. Let $P \times Q$ be any basic open set in $X \times Y$ containing (x, z). Therefore, by the above argument, $F^-(int\ clQ) \cap sclP \neq \emptyset$ and hence, $F^-(int\ clQ) \cap int\ clP \neq \emptyset$. Thus, $(int\ clP \times int\ clQ) \cap G(F) \neq \emptyset$, i.e., $int\ cl(P \times Q) \cap G(F) \neq \emptyset$. So, $(x, z) \in \delta{-}clG(F) = G(F)$ (as the graph of F i.e. $G(F)$ is δ-closed). Hence, $(x, z) \in [(\{x\} \times Y) \cap G(F)]$ so that $z \in \pi_2[(\{x\} \times Y) \cap G(F)] = F(x)$, where $\pi_2 : X \times Y \to Y$ is the second projection map. Therefore, $s(F, x) \subseteq F(x)$. But we always have $F(x) \subseteq s(F, x)$ for each $x \in X$. Hence, $s(F, x) = F(x)$ for each $x \in X$. □

Theorem 3.3. *If a multifunction $F : X \to Y$ from a space X into a space Y satisfies $s(F, x) = F(x)$ for each $x \in X$, then the graph $G(F)$ of F is semi-closed.*

Proof. If $(x, z) \notin G(f)$, then $z \notin F(x) = s(F, x)$. So, for some $U \in \tau(z)$ and some $V \in SO(x)$, $int\ clU \cap F(sclV) \neq \emptyset$, i.e., $(sclV \times sclU) \cap G(F) = \emptyset$.

But $scl(V \times U) \subset sclV \times sclU$ and $V \times U$ is a semi-open set in $X \times Y$ containing (x, z). So, $scl(V \times U) \cap G(F) = \emptyset$ and hence, $(x, z) \notin scl_\theta G(F)$. Therefore, $G(F)$ is semi-θ-closed. □

Corollary 3.4. *Let the spaces X and Y be such that every semi-θ-closed set is δ-closed in the product space $X \times Y$. Then for a multifunction $F : X \to Y$, the graph $G(F)$ is δ-closed if and only if $s(F, x) = F(x)$ for each $x \in X$.*

In Section 2, as an application of s-cluster sets of a function, we have characterized weakly Hausdorff spaces. In this section, we give some new characterizations of s-closed spaces via s-cluster sets of multifunctions.

Definition 3.5. *For a multifunction $F : X \to Y$ and a subset B of X, the notion $s(F, B)$ means $\bigcup_{x \in B} s(F, x)$.*

Notation 3.6. *For any subset S of a topological space Z, we denote the closure (resp. semi-closure, semi-θ-closure, interior closure) of S by $cl_Z S$ (resp. $scl_Z S$, $scl_\theta^Z S$, $int_Z\ cl_Z S$).*

Theorem 3.7. *For any topological space X, the following are equivalent:*

(i) *X is s-closed.*

(ii) *For each multifunction $F : X \to Y$, where Y is any topological space, $\cap\{\delta - clF(U) : U \in SO(B)\} \subseteq s(F, B)$ for each semi-θ-closed set B of X.*

(iii) *For each multifunction $F : X \to Y$, where Y is any topological space, $\cap\{scl_\theta F(U) : U \in SO(B)\} \subseteq s(F, B)$ for each semi-θ-closed set B of X.*

Proof. (i) ⇒ (ii) If B is a semi-θ-closed set in an s-closed space X, then by the Proposition 4.2 of Maio and Noiri [7] (every semi-θ-closed subset of an s-closed space X is s-closed relative to X), B is s-closed relative to X. Now, let $y \in \cap\{\delta - clF(U) : U \in SO(B)\}$. Then $int\ clV \cap F(U) \neq \emptyset$, i.e., $F^-(int\ clV) \cap U \neq \emptyset$ for each $V \in \tau(y)$ and for each $U \in SO(B)$. Clearly, $\mathcal{F} = \{F^-(int\ clV) : V \in \tau(y)\}$ is a filter base on X which meets B. Since B is s-closed relative to X, therefore by the Proposition 4.1 of Maio and Noiri [7] (a subset A of a topological space (X, τ) is S-closed relative to X if and only if every filter base on X which meets A, SR-adheres at some point of A), $B \cap SR - ad\mathcal{F} \neq \emptyset$. Let $b \in B \cap SR - ad\mathcal{F}$. Then $b \in B$ and $sclW \cap F^-(int\ clV) \neq \emptyset$, i.e., $F(sclW) \cap int\ clV \neq \emptyset$ for each $W \in SO(b)$ and for each $V \in \tau(y)$. Hence, $y \in \delta - clF(sclW)$ for each $W \in SO(b)$. Therefore, $y \in s(F, b) \subseteq s(F, B)$.

(ii) ⇒ (iii) This follows from the hypothesis (ii) and from the fact that for any subset A of Y, $scl_\theta A \subset \delta - clA$.

(iii) $\Rightarrow$ (i) Let $\mathcal{G}$ be a filterbase on X, and also let $Y = X \cup \{p\}$ where $p \notin X$. Clearly, the family $\tau_Y = \{V \subseteq Y : p \notin V\} \cup \{V \subseteq Y : p \in V, G \subseteq V$ for some $G \in \mathcal{G}\}$ forms a topology on Y. Consider the inclusion function $f : X \to Y$, i.e., $f(x) = x$ for all $x \in X$. Since X is semi-θ-closed in X, by (iii), $s(f,X) \supseteq \cap\{scl_\theta^Y f(U) : U \in SO(X)\} = \cap\{scl_\theta^Y U : U \in SO(X)\} = scl_\theta^Y X$. We now prove the last equality. Let $S \in SO(p)$. Then there exists a $V \in \tau_Y$ such that $V \subseteq S \subseteq scl_Y(S) \subseteq cl_Y V$. If $p \notin V$, then $V \subseteq X$ and hence, $scl_Y V \cap X \neq \emptyset$. If $p \in V$, then there is some $G \in \mathcal{G}$ such that $G \subseteq V$ and hence, $scl_Y G \subseteq scl_Y V$. Thus, $X \cap scl_Y V \neq \emptyset$. So, in any case, $X \cap scl_Y V \neq \emptyset$. Since $scl_Y V \subseteq scl_Y S$, $X \cap scl_Y S \neq \emptyset$. Hence, $p \in scl_\theta^Y X$. So, $p \in s(f,x)$ for some $x \in X$. Let $W \in SO(x)$ and $G \in \mathcal{G}$. Since $Y - (G \cup \{p\})$ is an open set in (Y, τ_Y), $scl_Y(G \cup \{p\}) = G \cup \{p\}$. Now, $G \cap scl_X W = (G \cup \{p\}) \cap f(scl_X W) = scl_Y(G \cup \{p\}) \cap f(scl_X W) = int_Y cl_Y(G \cup \{p\}) \cap f(scl_X W) \neq \emptyset$ (as $p \in s(f,x)$ and $G \cup \{p\} \in \tau_Y$). Therefore, $x \in SR - ad\mathcal{G}$. Hence, by Proposition 3.1 of Maio and Noiri [7] (A topological space (X, τ) is s-closed if and only if every filter base on X SR-adheres at some point of X), X is s-closed. □

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