UNCOUNTABLY MANY MUTUALLY DISJOINT, DENSE AND CONVEX SUBSETS OF ℓ^2 WITH APPLICATIONS TO PATH CONNECTED SUBSETS OF SPHERES

SAMUEL H. CRESWELL

ABSTRACT. Hyperplanes in $\mathbb{R}^n$ are extended to affine subspaces of ℓ^2, independently of the Axiom of Choice. These affine subspaces form a set of uncountably many mutually disjoint, dense and convex subsets of ℓ^2. A homeomorphism maps ℓ^2 to the sum of these sets. Further, any sphere S in ℓ^2 contains an uncountable collection of mutually disjoint and path connected subsets, each of which is dense in S.

1. INTRODUCTION

If $p \geq 1$, then we define ℓ^p as the space of sequences of real numbers, as follows

$$\ell^p \stackrel{\text{def}}{=} \left\{ x = (x_i) \,\middle|\, i > 0, x_i \in \mathbb{R}, \sum |x_i|^p < \infty \right\}.$$

If $x, y \in \ell^p$, then the distance between x and y is defined by

$$\|x - y\|_p \stackrel{\text{def}}{=} \left(\sum |x_i - y_i|^p \right)^{1/p}.$$

Throughout this paper $\|\cdot\|$ shall mean $\|\cdot\|_2$. It is straightforward to verify that ℓ^2, equipped with this distance function, is a complete metric space. Further details are available in [4].

The geometry of infinite dimensional spaces differs markedly from that of finite dimensional spaces. We investigate the geometry of ℓ^2, which is isometric to any Hilbert space. First, we show that there exists an uncountable, mutually disjoint collection of affine subspaces of ℓ^2, each dense and convex in ℓ^2. We then display a function which maps ℓ^2 homeomorphically to the union of the affine subspaces.

We then use these affine subspaces as a tool to explore the geometry of ℓ^2. In particular, we show that each sphere in ℓ^2 contains an uncountable collection of mutually disjoint, path connected subsets, each of which is dense in the sphere.

V. Klee [1] investigated uncountable collections of mutually disjoint, dense and convex subsets of normed linear spaces. Explicitly, Klee's study of the convex and dense subsets of Banach Spaces utilized Hamel bases, which are based upon the Axiom of Choice. Ženíšek [5] and Tukey [3] also developed results concerning convex and dense sets in infinite dimensional spaces which are independent of our results. Our work is independent of these papers in respect to methodology and results and does not use the Axiom of Choice.

2. AFFINE SUBSPACES IN ℓ^2

Definition 1. *For $c \in \mathbb{R}$, define X_c to be the set of all real-valued sequences $x = (x_i) \in \ell^1$ such that $\sum x_i = c$.*

Further, note three properties of X_c:

(1) As $\ell^1 \subset \ell^2$, $X_c \subset \ell^2$.
(2) Define $f(x) = \sum x_i$ which is a bounded linear functional on ℓ^1. Thus $X_c = f^{-1}(c)$ is closed in ℓ^1.
(3) $\bigcup_{c \in \mathbb{R}} X_c = \ell^1$.

Such subspaces are defined similarly in finite dimensions. Unlike the finite dimensional case, we have the following lemma.

Lemma 1. *For each $c \in \mathbb{R}$, X_c is dense in ℓ^2.*

Proof. Let $x = (x_i) \in \ell^2$ and $\epsilon > 0$ be arbitrary. We will construct a point $z \in X_c$ such that $\|x - z\| < \epsilon$ by altering the tail of x. For natural numbers N and M we have

$$\begin{aligned} x^{[N]} &\stackrel{\text{def}}{=} (x_1, x_2, \ldots, x_N, 0, 0, \ldots), \\ s &\stackrel{\text{def}}{=} x_1 + x_2 + \cdots + x_N, \\ y &\stackrel{\text{def}}{=} \left(\underbrace{0, 0, \ldots, 0}_{N \text{ terms}}, \underbrace{\frac{c-s}{M}, \frac{c-s}{M}, \ldots, \frac{c-s}{M}}_{M \text{ terms}}, 0, 0, \ldots \right) \quad \text{and} \\ z &\stackrel{\text{def}}{=} x^{[N]} + y. \end{aligned}$$

Fix N large enough so that $\left\|x - x^{[N]}\right\| < \epsilon/2$. Next, fix M large enough so that $\|y\| < \epsilon/2$. Plainly, $z = (z_i)$ is both square and absolutely summable since at most finitely many terms are nonzero. Further, $\sum z_i = s + M\left((c-s)/M\right) = c$, and so $z \in X_c$. Finally

$$\begin{aligned} \|x - z\| &\leq \left\|x - \left(x^{[N]} + y\right)\right\| = \left\|\left(x - x^{[N]}\right) - y\right\| \\ &\leq \left\|x - x^{[N]}\right\| + \|y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Lemma 2. *For each $c \in \mathbb{R}$, X_c is convex in ℓ^2.*

Proof. Let $x, y \in X_c$ and $t \in \mathbb{R}$. We will show that $z \stackrel{\text{def}}{=} tx + (1-t)\,y \in X_c$. By the triangle inequality

$$\sum |z_i| = \sum |tx_i + (1-t)\,y_i| \le |t| \sum |x_i| + |1-t| \sum |y_i| < \infty$$

since elements of X_c are absolutely summable by definition. Thus, $z = (z_i)$ is absolutely summable, and hence is square summable and summable. Finally

$$\sum z_i = \sum (tx_i + (1-t)\,y_i) = t \sum x_i + (1-t) \sum y_i = tc + (1-t)\,c = c.$$

Therefore, $z \in X_c$ as desired. □

Remark 1. *The real number t in the proof of the prior lemma is not restricted to $[0,1]$. It follows that each X_c is an affine subspace of ℓ^2 (namely a translate of the linear subspace X_0).*

Remark 2. *If $c \in \mathbb{R}$, then $X_c \subset \ell^1$ is not dense in ℓ^1. In contrast, Lemma 1 showed that X_c is dense in ℓ^2.*

Remark 3. *Of course, affine subspaces of $\mathbb{R}^n$ are convex. However, $\mathbb{R}^n$ is the only convex and dense subset of $\mathbb{R}^n$; several proofs of this fact may be given using continuity of the determinant and so forth. It is instructive to see where such proofs fail in ℓ^2. In ℓ^2 the product of the limits of an infinite degree polynomial is not necessarily the limit of its products. Accordingly, solutions of finite subsystems involving the use of a determinant may not be reliably extended to solutions of systems of infinitely many equations with infinitely many unknowns in ℓ^2 by passing to a limit* [4, p. 3].

Theorem 1. *The set $K \stackrel{\text{def}}{=} \{X_c \,|\, c \in \mathbb{R}\}$ is an uncountable collection of mutually disjoint, dense and convex subsets of ℓ^2.*

Proof. By the definition of X_c, we see that $X_c = X_d$ if and only if $c = d$. Therefore $c \mapsto X_c$ is a bijection $\mathbb{R} \to K$ showing that the cardinality of K equals the cardinality of $\mathbb{R}$. Lemmas 1 and 2 complete the proof. □

Recall that for a real number c, the sign of c, denoted $\sigma(c)$, is defined by

$$\sigma(c) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } c > 0, \\ 0, & \text{if } c = 0 \text{ and} \\ -1, & \text{if } c < 0. \end{cases}$$

Define the nonlinear function T with domain ℓ^2 by

$$T(x) \stackrel{\text{def}}{=} \left(\sigma(x_1)x_1^2, \sigma(x_2)x_2^2, \sigma(x_3)x_3^2, \ldots\right). \tag{1}$$

Note that as $\bigcup_{c\in\mathbb{R}} X_c = \ell^1$, $\bigcup_{c\in\mathbb{R}} X_c$ is a linear subspace of ℓ^2.

Lemma 3. *The function T is a bijection $\ell^2 \to \ell^1 = \bigcup_{c\in\mathbb{R}} X_c$.*

Proof. Let $x \in \ell^2$. As x is square summable, $T(x)$ is absolutely summable and hence square summable and summable. Therefore, $T(x) \in X_c$ where $c = \sum T(x)_i$. It remains to show that T is one-to-one and onto $\bigcup_{c\in\mathbb{R}} X_c$.

If $T(x) = T(y)$, then $\sigma(x_i)\, x_i^2 = \sigma(y_i)\, y_i^2$ for all $i \in \mathbb{N}$. This implies $x_i = y_i$ for all $i \in \mathbb{N}$, and so $x = y$ and T is one-to-one. In particular, T^{-1} exists.

Let $y = (y_i) \in X_c$ for some $c \in \mathbb{R}$. We claim that

$$T^{-1}(y) = \left(\sigma(y_1)\sqrt{|y_1|}, \sigma(y_2)\sqrt{|y_2|}, \sigma(y_3)\sqrt{|y_3|}, \dots\right) \in \ell^2. \tag{2}$$

Plainly T maps this element to y. As $y \in X_c$, y is absolutely summable. Hence, $T^{-1}(y)$ is square summable and lies in ℓ^2. Therefore, T is onto $\bigcup_{c\in\mathbb{R}} X_c$ and the lemma is proved. □

Due to a theorem of S. Mazur [2], T is a homeomorphism. Summarizing we have the following theorem.

Theorem 2. *The function $T : \ell^2 \to \ell^1 = \bigcup_{c\in\mathbb{R}} X_c$ defined by equation (1) is a homeomorphism whose image is the disjoint union of uncountably many dense and convex subsets of ℓ^2.*

3. Spheres in ℓ^2

Definition 2. *Let $x \in \ell^2$ and $r > 0$. The closed disk with center x and radius r is $D_{x,r} \overset{def}{=} \{y \in \ell^2 \mid \|x - y\| \le r\}$. The open disk with center x and radius r is $\{y \in \ell^2 \mid \|x - y\| < r\}$, which is the interior of $D_{x,r}$ and is denoted $\mathrm{Int}D_{x,r}$. The sphere with center x and radius r is $S_{x,r} \overset{def}{=} \{y \in \ell^2 \mid \|x - y\| = r\}$.*

The triangle inequality implies that $D_{x,r}$ is convex and closed in ℓ^2 (its complement is open), and that $\mathrm{Int}D_{x,r}$ is convex and open. A sphere $S_{x,r}$ is closed in ℓ^2 since its complement is open.

Lemma 4. *Let $S = S_{x,r}$ be a sphere and $c \in \mathbb{R}$. Let $y \in S$ and $\epsilon > 0$. Then there exists $z \in S \cap X_c$ such that $\|y - z\| < \epsilon$. In particular, $S \cap X_c$ has infinite cardinality and is dense in S.*

Proof. Let $D = D_{x,r}$ and $D' = D_{y,\epsilon}$. As $\mathrm{Int}D \cap \mathrm{Int}D'$ is nonempty and open in ℓ^2, this intersection contains a point u of X_c by density (Lemma 1). Similarly, there is $v \in X_c$ that lies in the open set $\mathrm{Int}D' \cap \left(\ell^2 - D\right)$ (see Figure 1). Note that $\|x - u\| < r$ and $\|x - v\| > r$.

Define $f : [0,1] \to \mathbb{R}$ by $f(t) \overset{\text{def}}{=} \|x - (tu + (1-t)v)\|$, which is continuous since addition, scalar multiplication and the norm are continuous.

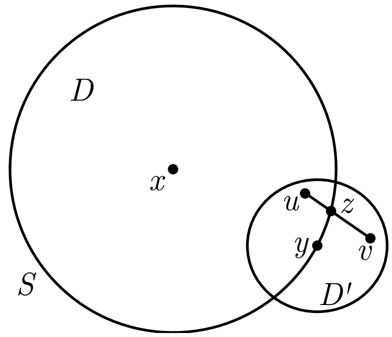

FIGURE 1. Obtaining $S_{x,r} \cap X_c$.

Now, $f(0) = \|x - v\| > r$ and $f(1) = \|x - u\| < r$. Thus, there is $t_0 \in (0, 1)$ such that $f(t_0) = r$. Let $z = t_0 u + (1 - t_0)v$, which lies in X_c by convexity (Lemma 2) and in S by construction. Finally, u and v lie in the convex open disk $\mathrm{Int}D'$, hence $z \in \mathrm{Int}D'$ and $\|y - z\| < \epsilon$, as desired. □

Remark 4. *The previous lemma has the following curious interpretation. Let $x \in X_0$ and $S = S_{x,r}$ be a sphere. Let $y \in S$, which may be viewed as a 'direction' from x. Let $\epsilon > 0$. By the previous lemma, there exist $p \in S \cap X_1$ and $q \in S \cap X_{10^6}$ so that $\|y - p\| < \epsilon$ and $\|y - q\| < \epsilon$ (that is, the directions p and q are ϵ-close to y). The linear path from x to p intersects, in order, the affine subspaces X_d where $0 \le d \le 1$. The linear path from x to q intersects, in order, the affine subspaces X_e where $0 \le e \le 10^6$. That is, a linear path arbitrarily close to a particular direction may intersect an arbitrarily large interval worth of affine subspaces. One may even choose $q \in X_{-1}$. Then, the linear path from x to q traverses, in reverse order, the affine subspaces from X_0 to X_{-1}. Notice the total disregard of accuracy of either direction or tolerance of direction of travel through a given set of affine subspaces.*

Lemma 5. *Let $S = S_{x,r}$ be a sphere in ℓ^2, $y \in S$ and $\epsilon > 0$. Then there exists $z \in S$ such that $\|y - z\| < \epsilon$ and z is not summable.*

Proof. Let $D = D_{x,r}$ and $D' = D_{y,\epsilon}$. Define $h \overset{\text{def}}{=} (1, 1/2, 1/3, 1/4, \ldots) \in \ell^2$, which is not summable. It is well-known that $\|h\| = \pi/\sqrt{6}$. By the density of X_1 in ℓ^2 (Lemma 1), there are points $p, q \in X_1$ so that $p \in \mathrm{Int}D \cap \mathrm{Int}D'$ and $q \in \mathrm{Int}D' \cap \left(\ell^2 - D\right)$ (see Figure 2). The idea now is to perturb p and q by a small scalar multiple of h and then take a convex combination of the

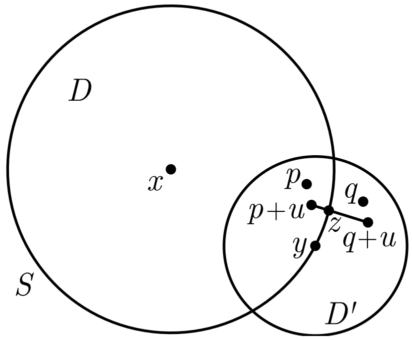

FIGURE 2. Nonsummable point in $S_{x,r} \cap D'$.

resulting vectors that lie in S. Let

$$\delta = \frac{\sqrt{6}}{\pi} \min \{r - \|p - x\|, \epsilon - \|p - y\|, \|q - x\| - r, \epsilon - \|q - y\|\}$$

and define $u \stackrel{\text{def}}{=} \delta h$. Then by the triangle inequality $p + u \in \mathrm{Int} D \cap \mathrm{Int} D'$ and $q + u \in \mathrm{Int} D' \cap \big(\ell^2 - D\big)$. A continuity argument, as in the proof of the previous lemma, yields a convex combination $z \stackrel{\text{def}}{=} t(p+u) + (1-t)(q+u)$, where $t \in (0, 1)$, so that $\|x - z\| = r$ (that is, $z \in S$). As $\mathrm{Int} D'$ is convex, $z \in \mathrm{Int} D'$ and so $\|y - z\| < \epsilon$. Finally, z is not summable since $z = tp + (1-t)q + u$, p and q are absolutely summable with sum 1, $u = \delta h$ with $\delta > 0$ and h is not summable. □

Corollary 1. *Let $S = S_{x,r}$ be a sphere and $c \in \mathbb{R}$. Then the intersection $S \cap X_c$ is not closed.*

Proof. By the previous lemma, there is a nonsummable point $y \in S$. By Lemma 4 there is a sequence of points in $S \cap X_c$ converging to y. However, $y \notin X_c$ since points in X_c are summable. □

Let C denote the set of $x \in \ell^2$ such that x is not summable.

Corollary 2. *Let $S = S_{x,r}$ be a sphere in ℓ^2. Then $S \cap C$ is not closed.*

Proof. By Lemma 4 there is $y \in X_1 \cap S$. By Lemma 5, there is a sequence of nonsummable points in S converging to the (absolutely) summable point y. The result follows. □

Lemma 6. *Fix a sphere $S = S_{x,r}$ and let $D = D_{x,r}$. Let $p \in \mathrm{Int} D$ and $q \in \ell^2 - D$. Let L denote the line in ℓ^2 containing p and q. Then*

(i) *there is a unique point $y \in L \cap \mathrm{Int} D$ such that $\langle y - x, q - p\rangle = 0$,*
(ii) *$y = y(p)$ is a continuous function of p,*

(iii) *there is a unique* $\tau \in (0,1)$ *such that* $\|y + \tau(q-y) - x\| = r$ *and*
(iv) $\tau = \tau(p)$ *is a continuous function of* p.

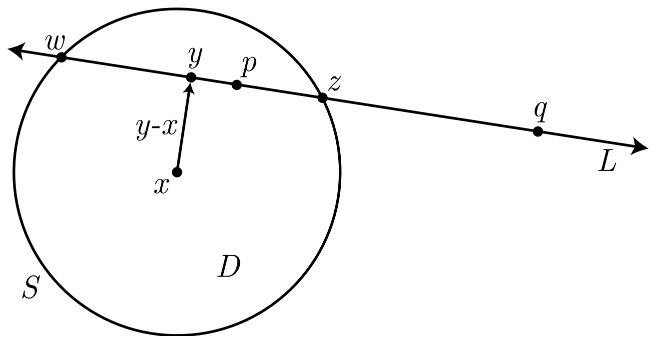

FIGURE 3. Projecting p to S.

Proof. Figure 3 shows the setup of the lemma. The point y will have the form $y = p + t(q-p)$ for some $t \in \mathbb{R}$. To find y we solve

$$\langle p + t(q-p) - x, q-p\rangle = 0$$

for t and obtain

$$t = \frac{\langle x-p, q-p\rangle}{\|q-p\|^2},$$

which is our choice of t. With x and q fixed, $t = t(p)$ is a continuous function of p since the inner product is continuous and the denominator of t does not vanish. We claim that $\|y-x\| \leq \|p-x\|$ (in fact, y is the point of L closest to x). To see this, we compute

$$\begin{aligned}
\|y-x\|^2 &= \langle p + t(q-p) - x, p + t(q-p) - x\rangle \\
&= \langle p-x, p-x\rangle + t^2\langle q-p, q-p\rangle + 2t\langle p-x, q-p\rangle \\
&= \|p-x\|^2 + \frac{\langle p-x, q-p\rangle^2}{\|q-p\|^2} - 2\frac{\langle p-x, q-p\rangle^2}{\|q-p\|^2} \\
&= \|p-x\|^2 - \frac{\langle p-x, q-p\rangle^2}{\|q-p\|^2} \\
&\leq \|p-x\|^2,
\end{aligned}$$

where the third equality used our choice of t. The claim is proven. Therefore, $\|y-x\| \leq \|p-x\| < r$, and so $y \in \mathrm{Int}D$. We have proved conditions (i) and (ii).

Now we seek the point z in Figure 3, which will have the form $z = y + \tau(q-y)$ for some $\tau \in \mathbb{R}$. We must solve for τ in

$$\|y + \tau(q-y) - x\| = r.$$

Square both sides and expand to obtain

$$\|y-x\|^2 + \tau^2 \|q-y\|^2 + 2t\langle y-x, q-y\rangle = r^2.$$

The inner product in this equation vanishes by condition (i) and the definition of y. Thus, we obtain

$$\tau = \pm \frac{\sqrt{r^2 - \|y-x\|^2}}{\|q-y\|}$$

where $r^2 - \|y-x\|^2 > 0$ since $y \in \mathrm{Int}D$. Therefore, $L \cap S$ consists of exactly two points z and w, where we choose z to correspond to $\tau > 0$ and w to correspond to $\tau < 0$. Adopting the convention that $a * b * c$ means a, b, and c are collinear and that b is between a and c, the definitions of z and w imply $w * y * z * q$ (this is used in Remark 5). It is easy to verify that $|\tau| < 1$ using the Pythagorean theorem. We choose $\tau > 0$. Then, with x and q fixed, $\tau = \tau(p)$ is a function of p since $y = y(p)$, and $\tau = \tau(p)$ is continuous since the inner product is continuous and the denominator of τ does not vanish. We have proved conditions (iii) and (iv), and the lemma follows. □

Recall that a nontrivial continuum is a nonconstant continuous image of $[0,1]$.

Corollary 3. *Let $S = S_{x,r}$ be a sphere and $c \in \mathbb{R}$. Then $S \cap X_c$ contains a nontrivial continuum.*

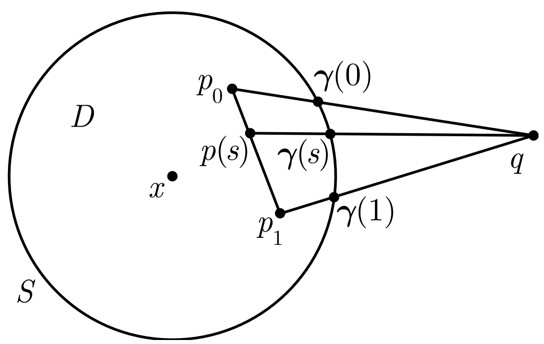

FIGURE 4. Projecting an arc to S.

Proof. Let $D = D_{x,r}$. By the density of X_c in ℓ^2 (Lemma 1), there are distinct points p_0 and p_1 in $X_c \cap \mathrm{Int}D$. Choose any point $q \in X_c - D$ so that q does not lie on the line containing p_0 and p_1 (use the density of X_c). Figure 4 shows the current setup.

Define $p : [0, 1] \to \mathrm{Int}D$ by

$$p(s) \stackrel{\text{def}}{=} p_0 + s\,(p_1 - p_0)\,,$$

which is continuous since addition and scalar multiplication are continuous. Notice that $\|x - p(s)\| < r$ for all $s \in [0, 1]$ by the triangle inequality. Thus, p maps continuously into $\mathrm{Int}D$.

Define $\gamma : [0, 1] \to S$ by

$$\gamma(s) \stackrel{\text{def}}{=} y\,(p(s)) + \tau\,(p(s))\,(q - y\,(p(s)))$$

where y and τ are given by the previous lemma. The image of γ lies in S by condition (iii) of the previous lemma and in X_c since X_c is convex (Lemma 2). The composition of continuous functions is continuous, from which we see γ is continuous. Finally, we claim that γ is one-to-one. But this is immediate, since otherwise p_0, p_1 and q would be collinear, contradicting our choice of q. $\square$

Remark 5. *We now proceed to strengthen the hypothesis of Lemma 6 in two ways: first we demonstrate that the lemma holds if $p \in S$, then we demonstrate that the lemma holds for any $q \in \ell^2$ satisfying $\langle q - p, x - p\rangle < 0$. Let us examine the proof of Lemma 6 and make some observations along the way. We obtain the same unique $t \in \mathbb{R}$ since $\|q - p\| > 0$, and $t = t(p)$ is continuous. The displayed computation shows that $\|y - x\| \le \|p - x\| = r$ (with equality if and only if $\langle p - x, q - p\rangle = 0$), so we conclude that $y \in L \cap D$. In particular, $y \in S$ if and only if $\langle p - x, q - p\rangle = 0$. The argument for τ yields two points (or a single point if and only if $\tau = 0$) in $L \cap S$. Once a choice of sign for τ is made, then clearly $\tau = \tau(p)$ is continuous. With these minor changes, Lemma 6 holds for $p \in S$. Let us pause and look at the possibility $y \in S$. The following are equivalent: $y \in S$, $\langle p - x, q - p\rangle = 0$, $t = 0$, $y = p$, $\tau = 0$, and $L \cap S$ is a single point (namely p). Equivalence of the first two was observed earlier in this remark, and the rest is clear by the definitions of t, y and τ in Lemma 6.*

Let us now restrict to the case where $p \in S$ and $\langle q - p, x - p\rangle < 0$. In particular, $L \cap S = \{z, w\}$ where $w * y * z * q$, z corresponds to $\tau > 0$ and w corresponds to $\tau < 0$. We claim that: (i) $\|q - x\| > \|p - x\| = r$ (so $q \in \ell^2 - D$) and (ii) $p = z$. Notice that

$$0 > \langle q - p, x - p\rangle = \langle (q - x) + (x - p), x - p\rangle$$

which implies that

$$\begin{aligned} 0 < \|x-p\|^2 &< \langle q-x, p-x\rangle \\ &= |\langle q-x, p-x\rangle| \leq \|q-x\| \, \|p-x\| \end{aligned}$$

by the Cauchy-Schwarz inequality, which proves (i). With $q \in \ell^2 - D$ *consider Figure 3 with* $p = z$ *or* $p = w$*. Remembering that* $\langle q-y, x-y\rangle = 0$ *(as in the proof of Lemma 6), we may determine whether* $p = z$ *or* $p = w$ *by verifying that*

$$\langle q - (y + \tau(q-y)), x - (y + \tau(q-y))\rangle = -\tau(1-\tau)\|q-y\|^2 . \quad (3)$$

As $\langle q-p, x-p\rangle < 0$*, the inner product in equation* (3) *is negative, which assures that* $0 < \tau < 1$*. Thus* $p = z$*, proving (ii).*

Corollary 4. *Let* $S = S_{x,r}$ *be a sphere and* $c \in \mathbb{R}$*. Let* p_0 *and* p_1 *be distinct, non-antipodal points in* $S \cap X_c$*. Then there is a path in* $S \cap X_c$ *with endpoints* p_0 *and* p_1*.*

Proof. Let p_0 and p_1 be distinct, non-antipodal points in $S \cap X_c$, which exist by density (Lemma 4). The idea is to pick an appropriate q and then to project exactly as in Corollary 3 (see Figure 4) except with p_0 and p_1 on S. By the beginning of Remark 5, the projection γ exists and is continuous, although care must be taken to ensure that p_0 and p_1 are the endpoints of the resulting path in $S \cap X_c$. Using the observation near the end of Remark 5, it suffices to demonstrate the existence of a point $q \in X_c$ so that

$$\langle q-p_0, x-p_0\rangle < 0 \quad \text{and} \quad \langle q-p_1, x-p_1\rangle < 0 \quad (4)$$

assuring that q is not on the line through p_0 and p_1, as required by the proof of Corollary 3. Thus $\tau > 0$ is the correct choice for both endpoints. Let $m = (p_0 + p_1)/2$ be the midpoint of $\overline{p_0p_1}$, which is distinct from x since p_0 and p_1 are not antipodal points of S. Let $Q = x + \lambda(m-x)$ for $\lambda \in \mathbb{R}$ to be determined. Given that

$$\langle p_0 - x, p_0 - x\rangle = r^2 = \langle p_1 - x, p_1 - x\rangle$$

it follows easily that

$$\langle m-x, x-p_0\rangle = \langle m-x, x-p_1\rangle .$$

We compute and use the previous equation to obtain

$$\begin{aligned} \langle Q-p_0, x-p_0\rangle &= r^2 + \lambda\langle m-x, x-p_0\rangle \\ &= r^2 + \lambda\langle m-x, x-p_1\rangle \\ &= \langle Q-p_1, x-p_1\rangle . \end{aligned}$$

Thus, for sufficiently large $\lambda > 0$, condition (4) will be satisfied by Q, provided we show that $\langle m - x, x - p_0 \rangle < 0$, or equivalently $\langle m - x, p_0 - x \rangle > 0$. We have

$$\begin{aligned}\langle m - x, p_0 - x \rangle &= \langle (m - p_0) + (p_0 - x), p_0 - x \rangle \\ &= \langle m - p_0, p_0 - x \rangle + r^2\end{aligned}$$

and the result follows since

$$|\langle m - p_0, p_0 - x \rangle| \leq \|m - p_0\| \, \|p_0 - x\| < r \cdot r$$

by the Cauchy-Schwarz inequality and since $\|p_1 - p_0\| < 2r$. Finally, Q need not lie in X_c, however, by density of X_c in ℓ^2 (Lemma 1) and continuity of addition, scalar multiplication and the inner product, there is $q \in X_c$ sufficiently close to Q so that q satisfies condition (4). $\square$

Remark 6. *The previous corollary holds even when p_0 and p_1 are antipodal points on S by applying the corollary twice. Simply choose any third point $p_2 \in S \cap X_c$, which is clearly not antipodal to p_0 or p_1, and concatenate paths from p_0 to p_2 and from p_2 to p_1 provided by the corollary.*

Theorem 3. *Let $S = S_{x,r}$ be a sphere in ℓ^2. Then S contains an uncountable collection of mutually disjoint and path connected subsets, each of which is dense in S.*

Proof. Theorem 1 showed that $K = \{X_c \,|c \in \mathbb{R}\}$ is an uncountable collection of mutually disjoint, dense and convex subsets of ℓ^2. Consider the set $J \stackrel{\text{def}}{=} \{S \cap X_c \,|c \in \mathbb{R}\}$. Each of the intersections $S \cap X_c$ is dense in S by Lemma 4 and is path connected by Corollary 4 and Remark 6. As the affine subspaces are mutually disjoint, so are their intersections with S. The cardinality of J equals the cardinality of K and the result follows. $\square$

Remark 7. *One may obtain k-dimensional disks (simplices) in $S_{x,r} \cap X_c$ by choosing $p_0, p_1, \ldots, p_k$ in $X_c \cap \mathrm{Int}D$ and $q \in X_c - D$ in general position (meaning $p_0, p_1, \ldots, p_k, q$ do not lie in a k-dimensional affine subspace). The domain of p is the k-simplex Δ_k in $\mathbb{R}^k$ that is the convex hull of $\overrightarrow{0}, e_1, e_2, \ldots, e_k$ where e_i is the standard basis vector whose ith coordinate is 1 and whose other coordinates are 0. Naturally, p sends a convex combination of $\overrightarrow{0}, e_1, e_2, \ldots, e_k$ to the corresponding convex combination of $p_0, p_1, \ldots, p_k$, which is continuous (addition and multiplication are continuous) and one-to-one (by the general position above). The domain of γ is also Δ_k and otherwise its definition remains the same. As above, γ maps into S, is continuous, and is one-to-one (by the general position).*

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Mathematics Department, San Antonio College, San Antonio, Texas 78212-4299

E-mail address: screswell@alamo.edu