A CONTINUOUS BIJECTION FROM ℓ^2 ONTO A SUBSET OF ℓ^2 WHOSE INVERSE IS EVERYWHERE UNBOUNDEDLY DISCONTINUOUS, WITH AN APPLICATION TO PACKING OF BALLS IN ℓ^2

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Abstract. There is a continuous bijection from ℓ^2 onto a subset of ℓ^2 whose inverse is everywhere unboundedly discontinuous. If B is a ball in ℓ^2, then the continuous bijection defined on ℓ^2 maps countably many mutually disjoint balls of ℓ^2 into countably many mutually disjoint balls in B, making those images mutually disjoint.

1. Introduction

This work is a sequel to the author's paper [1] in which the author showed that there is a continuous bijection T, to be defined below, from ℓ^2 onto a subset of ℓ^2 such that T^{-1} is everywhere discontinuous. Background information concerning continuous bijections with discontinuous inverses is provided in [1].

In this paper, we will show that the inverse is even more pathological, namely *everywhere unboundedly discontinuous*, a term which we will define below.

Further, as an application, we shall show that if B is a ball in ℓ^2, then the continuous bijection T defined on ℓ^2 maps countably many mutually disjoint balls of ℓ^2 into countably many mutually disjoint balls in B, making those images mutually disjoint, by the invertibility of T.

2. Elementary Properties of ℓ^1 and ℓ^2

This section, included for completeness, recalls basic properties of ℓ^2 sufficient for our uses. See [3, Chapter 1] for a good introduction to Hilbert spaces and for proofs of the facts stated below.

We will denote a sequence $(x_1, x_2, x_3, \ldots)$ of real numbers by (x_i) or simply x. By definition

$$\ell^2 \stackrel{\text{def}}{=} \left\{ x = (x_i) \,\middle|\, \sum x_i^2 < \infty \right\},$$

and

$$\ell^1 \stackrel{\text{def}}{=} \left\{ x = (x_i) \,\middle|\, \sum |x_i| < \infty \right\}.$$

Note that whenever limits are suppressed in a sum, they are assumed to run over the natural numbers $1, 2, 3, \ldots$. In ℓ^2 addition and scalar multiplication are defined componentwise: $x + y \stackrel{\text{def}}{=} (x_i + y_i)$ and $cx \stackrel{\text{def}}{=} (cx_i)$ for $c \in \mathbb{R}$. Clearly ℓ^2 is closed under scalar multiplication. For $x, y \in \ell^2$ define the inner product of x and y by

$$\langle x, y \rangle \stackrel{\text{def}}{=} \sum x_i y_i$$

and the norm of x by

$$\|x\| \stackrel{\text{def}}{=} \langle x, x \rangle^{1/2} = \left(\sum x_i^2 \right)^{1/2}.$$

In [1, Lemmas 1 and 2] we showed for the function $T\colon \ell^2 \to \ell^1$, defined below, that T is a continuous bijection of ℓ^2 onto ℓ^1 where ℓ^2 is the metric space (ℓ^2, d) with $d(x, y) = \|x - y\|$ and ℓ^1 is considered a subspace of (ℓ^2, d). We denote the norm in ℓ^2 by $\|\cdot\|$ in every instance below.

Definition 1. *If f is a function defined on a subset $D(f)$ of ℓ^2 such that if $K, \delta > 0$ and $y \in D(f)$, there is a point $z \in D(f)$ such that $\|y - z\| < \delta$ and $\|f(y) - f(z)\| > K$, then f is everywhere unboundedly discontinuous.*

Perhaps an example of an everywhere unboundedly discontinuous function defined on [0, 1] would be helpful. Let $f\colon [0, 1] \to \mathbb{R}$ be defined as follows:

$$f(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ Q & \text{if } P \text{ and } Q \text{ are relatively prime positive integers such that} \\ & x = P/Q. \end{cases}$$

Allowing P to run over the odd positive integers and Q to run over positive integral powers of 2, $\{P/Q\}$ is dense in $[0, 1]$, from which it follows that f is everywhere unboundedly discontinuous. We now turn our attention to ℓ^2.

For $c \in \mathbb{R}$, we define $\sigma(c)$, by

$$\sigma(c) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } c \geq 0, \\ -1, & \text{if } c < 0. \end{cases}$$

Define the nonlinear function T with domain ℓ^2 by

$$T(x) \stackrel{\text{def}}{=} \left(\sigma(x_1)x_1^2, \sigma(x_2)x_2^2, \sigma(x_3)x_3^2, \ldots \right). \tag{1}$$

It may be verified that $T^{-1}\colon \ell^1 \to \ell^2$ is given by

$$T^{-1}(y) = \left(\sigma(y_1)\sqrt{|y_1|}, \sigma(y_2)\sqrt{|y_2|}, \sigma(y_3)\sqrt{|y_3|}, \dots\right) \in \ell^2. \tag{2}$$

Lemma 1. *The function T^{-1} is everywhere unboundedly discontinuous under the topology of ℓ^2.*

Proof. Let $y \in T\left(\ell^2\right)$ and $\epsilon, K > 0$. We will exhibit a point $z \in T\left(\ell^2\right)$ such that $\|y - z\| < \epsilon$ and $\left\|T^{-1}(y) - T^{-1}(z)\right\| > K$. For convenience, we set $M = 2K$. Note that $D(T^{-1}) = T(\ell^2) = \ell^1 \subset \ell^2$.

Fix $n \in \mathbb{N}$ so that $\dfrac{M}{\sqrt{n}} < \epsilon$. Define the real function

$$g(t) \stackrel{\text{def}}{=} \sqrt{|t| + \frac{M^2}{n}} - \sqrt{|t|}$$

which is continuous for all $t \in \mathbb{R}$. Hence,

$$g(t) \to \frac{M}{\sqrt{n}} \text{ as } t \to 0.$$

Therefore, there exists $\delta > 0$ so that

$$g(t) > \frac{M}{2\sqrt{n}} \text{ for all } |t| < \delta. \tag{3}$$

As $y \in T\left(\ell^2\right)$, $y = (y_i)$ is absolutely summable. Therefore, there exists $N \in \mathbb{N}$ so that

$$|y_i| < \delta \text{ for all } i > N. \tag{4}$$

Define

$$z \stackrel{\text{def}}{=} \begin{cases} y_i + \frac{\sigma(y_i)M}{n}, & \text{if } N+1 \le i \le N+n, \\ y_i, & \text{otherwise.} \end{cases}$$

As z differs from y in at most finitely many terms, $z \in T\left(\ell^2\right)$. Note that $\sigma\left(y_i\right) = \sigma\left(z_i\right)$ for all $i \in \mathbb{N}$. Also

$$\|y - z\|^2 = \sum_{i=N+1}^{N+n} \frac{\left(-\sigma\left(y_i\right)\right)^2 M^2}{n^2} = \frac{M^2}{n} < \epsilon^2$$

and so $\|y - z\| < \epsilon$. Finally,

$$\begin{aligned}\left\|T^{-1}(y)-T^{-1}(z)\right\|^2 &= \sum_{i=1}^{\infty}\left(\sigma\left(y_i\right)\sqrt{|y_i|}-\sigma\left(z_i\right)\sqrt{|z_i|}\right)^2\\ &= \sum_{i=N+1}^{N+n}\left(\sqrt{|y_i|}-\sqrt{|y_i|+\frac{M}{n}}\right)^2\\ &= \sum_{i=N+1}^{N+n}\left(\sqrt{|y_i|+\frac{M}{n}}-\sqrt{|y_i|}\right)^2\\ &> \sum_{i=N+1}^{N+n}\left(\frac{M}{2\sqrt{n}}\right)^2=\frac{M^2}{4},\end{aligned}$$

where the first equality used the definition of T^{-1} (Equation (2)), the second equality used $\sigma\left(y_i\right)=\sigma\left(z_i\right)$ for all i, and the inequality used (3) and (4). Thus,

$$\left\|T^{-1}(y)-T^{-1}(z)\right\| > \frac{M}{2}=K,$$

as desired. □

Summarizing we have the following theorem.

Theorem 1. *The function T considered in the topology of ℓ^2 and defined by Equation* (1) *is a continuous bijection (Lemmas 1 and 2,* [1]*) whose inverse T^{-1}, given by Equation* (2)*, is everywhere unboundedly discontinuous.*

We now state, as a corollary, an assertion which may be shown to be equivalent to Theorem 1. In the balance of this paper we shall make no distinction between the equivalent forms. Indeed, the second form of Theorem 1 will be used in the first paragraph of the next section.

Corollary 1. *If $x^{'}\in\ell^1$, there is a unbounded sequence $\{K_i\}_{i=1}^{\infty}$ of positive numbers and a sequence $\{z_i\}_{i=1}^{\infty}$ of points of ℓ^1 converging to $x^{'}$ such that if $i>0$, then $\left\|T^{-1}(x^{'})-T^{-1}(z_i)\right\|>K_i$.*

3. An Application to Packing of Balls in ℓ^2

We now restate Theorem 1 in a geometric form. Let $B=B_{x,r}$ be a ball in ℓ^2. As ℓ^1 is dense in ℓ^2 under the topology of ℓ^2 [2, Lemma 1], there is a ball $B^{'}\subset B$ whose center $x^{'}$ is a point of $\ell^1\subset\ell^2$ and whose radius $r^{'}<r$. Theorem 1 assures that there is a sequence $\{z_i\}_{i=1}^{\infty}$ of points of $B^{'}$ which converges to $x^{'}$ and an unbounded sequence $\{K_i\}_{i=1}^{\infty}$ of positive numbers such that for each $i>0$, $\left\|T^{-1}(x^{'})-T^{-1}(z_i)\right\|>K_i$. Choose a subsequence

$\{y_i\}_{i=1}^{\infty}$ of $\{z_i\}_{i=1}^{\infty}$ such that for $i > 0$, $\left\|x^{'} - y_{i+1}\right\| < \left\|x^{'} - y_i\right\|$. That is, we have chosen a subsequence which converges monotonically to $x^{'}$.

We will now show that the members of $\{y_i\}_{i=1}^{\infty}$ may be placed in mutually disjoint balls, each of which is a subset of $B^{'} \subset B$. First note that the members of $\{y_i\}_{i=1}^{\infty}$ are points of concentric spheres, each with center $x^{'}$. Also, as the spheres are concentric, the distance between a point of one of the spheres and a point of another one of these spheres is at least as great as the difference the radii of the two spheres. Let $r_i \stackrel{def}{=} \left\|x^{'} - y_i\right\|$ and $S_i \stackrel{def}{=} S_{x^{'},r_i}$. For $i > 0$, let $\rho_i < \dfrac{\min(r_{i-1} - r_i,\, r_i - r_{i+1})}{2}$ and $B_i^{'} \stackrel{def}{=} B_{y_i,\rho_i}$. If follows that if $i \neq j$, then $B_i^{'}$ and $B_j^{'}$ are mutually disjoint. That is, $\{B_i^{'}\}_{i=1}^{\infty}$ is a collection of mutually disjoint balls in $B^{'} \subset B$, which for each $i > 0$, are centered at y_i.

We now wish to assure that the inverse images under T of $\{y_i\}_{i=1}^{\infty}$ may be placed in mutually disjoint balls in ℓ^2. As for all $i > 0$,

$$\left\|T^{-1}(x^{'}) - T^{-1}(y_i)\right\| > K_i,$$

the triangular inequality assures that $\left\|T^{-1}(x^{'})\right\| + \left\|T^{-1}(y_i)\right\| > K_i$. As $\left\|T^{-1}(x^{'})\right\|$ is fixed and $\{K_i\}_{i=1}^{\infty}$ is an unbounded sequence of positive numbers, it follows that $\{\left\|T^{-1}(y_i)\right\|\}_{i=1}^{\infty}$ is unbounded above. Accordingly, there is a subsequence $\{w_i\}_{i=1}^{\infty}$ of $\{y_i\}_{i=1}^{\infty}$ such that for $i > 0$,

$$\left\|T^{-1}(w_{i+1})\right\| > 1 + \left\|T^{-1}(w_i)\right\|.$$

Noting that $T^{-1}(w_{i+1})$ and $T^{-1}(w_i)$ are points of concentric spheres centered at the origin, an argument in the second paragraph of this section assures that for $i > 0$, $\left\|T^{-1}(w_i) - T^{-1}(w_{i+1})\right\| > 1$.

As T is continuous from ℓ^2 onto $\ell^1 \subset \ell^2$, then for $i > 0$, T is continuous at $T^{-1}(w_i)$. Utilizing the observation in the last sentence of the preceding paragraph and the definition of continuity, there is an ϵ_i, with $0 < \epsilon_i < 1/2$, such that if $\beta_i \stackrel{def}{=} \beta_{T^{-1}(w_i),\epsilon_i}$, then $T(\beta_i) \subset B_i^{'}$. For $i \neq j$, β_i and β_j are mutually disjoint, as are $B_i^{'}$ and $B_j^{'}$. For each $i > 0$, $T(\beta_i) \subset B_i^{'} \subset B^{'} \subset B$. As T is invertible, T maps countably many mutually disjoint balls in ℓ^2 into countably many mutually disjoint balls of B, with all of these images being mutually disjoint. We may further note that the set of centers of these balls in the domain of the continuous bijection is unbounded. Finally, we have the following theorem.

Theorem 2. *There is a continuous bijection from ℓ^2 onto a subset of ℓ^2 which for each ball B in ℓ^2 maps countably many mutually disjoint balls* *of ℓ^2 into countably many mutually disjoint balls in B, making the images mutually disjoint. Further, the set of centers of the balls in domain of the continuous bijection is unbounded.*

Remark 1. *For any $i > 0$, $T(\beta_i)$ is not a convex body. Let q be a point of $T(\beta_i)$. As T maps ℓ^2 onto $\ell^1 \subset \ell^2$, $q \in \ell^1 \subset \ell^2$. Let $h \stackrel{def}{=} (1, 1/2, 1/3, \ldots)$. For any ϵ, $\|\epsilon h\| = \dfrac{|\epsilon|\pi}{\sqrt{6}}$. It follows that if v in ℓ^1, then $v + \epsilon h$ is in $\ell^2 - \ell^1$. Any ball centered at q has a radius r. By choosing $|\epsilon| < \dfrac{r\sqrt{6}}{\pi}$, that ball contains a point p of $\ell^2 - \ell^1$. If $T(\beta_i)$ were a convex body, then p would be a point of ℓ^1. Thus, $T(\beta_i)$ is not a convex body. That is, the 'packing' of balls is rather sparse.*

Definition 2. *Let $\{F\} \stackrel{def}{=} \{$sequences of rational numbers $f = (f_j) \mid$ for only finitely many j does $f_j \neq 0\}$.*

Remark 2. *Note that the ball B may be centered anywhere in ℓ^2 and have a radius as small as wished. That is, the set of balls B is dense in ℓ^2. Thus, Theorem 2 could have been altered to include the notion that there is a set of balls which are dense in ℓ^2 with the properties noted in Theorem 2. Specificially, allowing the centers of balls to range over $\{F\}$ and allowing the radii to range over $(1/2)^n$, produces a countable set of balls dense in ℓ^2. To avoid clutter we include that information as a Remark, rather than in Theorem 2.*

Remark 3. *Let us consider two balls B in ℓ^2. A ball B_i for one ball B may intersect a ball B_j for another ball B. Thus, there is a ball Γ which is a subset of $B_i \bigcap B_j$. As T is invertible, there is an apparent conundrum, as B_i and B_j contain images, $T(\beta_i)$ and $T(\beta_j)$, of two balls, β_i and β_j, that might be disjoint. Note that Remark 1 assures that neither $T(\beta_i)$ nor $T(\beta_j)$ is a convex body, thus assuring that the conundrum is not necessarily in conflict with the work herein. However, the exact process by which disjoint balls may be packed into balls centered at points of $\ell^1 \subset \ell^2$, without points common to the packings, is beyond the author's current understanding.*

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References

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