A NOTE ON ALMOST BAIRE BITOPOLOGICAL SPACES

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Abstract. In this paper, using bitopological semi-open sets, an asymmetric generalization of Haworth-McCoy's well-known theorem [4, Theorem 3.1] about Baire spaces is established.

The basic concepts of bitopological spaces (briefly, bispaces) were proposed in the fundamental work by J. C. Kelly [5]. Bispaces (i.e. ordered triples (X, τ_1, τ_2), where the base set X is equipped with topologies τ_1 and τ_2 which, speaking in general, are independent of each other) enable us to characterize the asymmetric topological properties which are outside the framework of classical topology. The methods of the bitopological theory are successfully applied in other contemporary mathematical disciplines such as mathematical analysis, lattice theory and so on.

Below the following notation is used: the interior and the closure of a set $A \subseteq X$ with respect to the topology τ_i are denoted by $\tau_i \mathrm{int} A$ and $\tau_i \mathrm{cl} A$, respectively, where $i \in \{1, 2\}$. If O is open with respect to τ_i, then we write $O \in \tau_i$. We denote by $\tau_i^* = \{A \cap U | U \in \tau_i\}$ the topology induced on the set A from the τ_i. The family of all i-open neighborhoods of a point $x \in X$ of the bispace (X, τ_1, τ_2) is denoted by $\sum_i^X(x)$, and the class of all i-dense subsets by $D_i(X)$.

Many important characterizations of Baire topological spaces are given by R. C. Haworth and R. A. McCoy [4]. Interesting results of the generalized Baire property in bispaces can be found in [1, 2, 3]. To avoid confusion, we use the definition of the almost Baire property given in [2]. In this note we establish one possible bitopological generalization of the topological fact [4, Theorem 3.1]: if a topological space (Y, γ) contains some countable set $W = \{y_n | \{y_n\} \in \gamma\}_{n=\overline{0;\infty}}$ and for every map $f\colon (X, \tau) \to (Y, \gamma)$ there exists a dense subset $A \subset X$ such that the restriction of f on the subspace (A, τ^*) is continuous, then (X, τ) is a Baire space.

In 1963, N. Levine [6] discovered, for topological spaces, the class of semi-open sets as a natural generalization of half-open intervals of reals (equipped with the usual topology). Subsequently, several topologists actively used such sets to describe the important topological notions of S-closed spaces, irresolute maps, extremally disconnected spaces and many others. Analogous classes of sets were introduced and studied for the bitopological theory in [7]. Such types of sets are intensively studied and applied in modifications of continuity by T. Noiri, H. Maki, T. Fukutake, V. Popa, etc. [3, 7]. A set A of a bispace (X, τ_1, τ_2) is called (i, j)-semi-open if $A \subset \tau_j \mathrm{cl} \tau_i \mathrm{int} A$ [2, 7]. A complement of an (i, j)-semi-open set is usually said to be (i, j)-semi-closed. The family of all (i, j)-semi-open subsets of X is denoted by $(i, j) - SO(X)$. It may be easily verified that if $A \in (i, j) - SO(X)$ and $B \in \tau_i$ then $A \cap B \in (i, j) - SO(X)$.

We remind the reader that a subset A of a bispace (X, τ_1, τ_2) is called (i, j)-nowhere dense if it satisfies the condition $\tau_i \mathrm{int} \tau_j \mathrm{cl} A = \emptyset$ where $i, j = 1, 2,\ i \neq j$. The class of all (i, j)-nowhere dense subsets of X is denoted by $(i, j) - ND(X)$. A set, when written in the form $A = \bigcup_{n=1}^{\infty} A_n$, where $\{A_n | A_n \in (i, j) - ND(X)\}_{n \in \mathbb{N}}$, is said to be of the (i, j)-first category (for brevity, we use the notation $A \in (i, j) - Catg_I(X)$). Otherwise, a set A is said to be of the (i, j)-second category and we write it in the form $A \in (i, j) - Catg_{II}(X)$.

Definition. *A bispace (X, τ_1, τ_2) is said to be almost (i, j)-Baire if any sequence of sets $\mathcal{F} = \{G_n | G_n \in \tau_j \cap D_i(X)\}_{n \in \mathbb{N}}$ satisfies the condition $\bigcap\limits_{n \in \mathbb{N}} G_n \in D_i(X)$, where $i, j = 1, 2,\ i \neq j$.*

It is known [2] that (X, τ_1, τ_2) is almost (i, j)-Baire if and only if $A \in (i, j) - Catg_I(X)$ implies $X \setminus A \in D_i(X)$.

Theorem. *Suppose a bispace (Y, γ_1, γ_2) contains a countable subset of distinct points $W = \{y_n | \{y_n\} \in \gamma_i\}_{n = \overline{0;\infty}}$. Further, assume that for any map $f \colon (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ there exists a set $A \in D_i(X) \cap (i, j) - SO(X)$ such that the restriction $f|_A \colon (A, \tau_i^*) \to (Y, \gamma_i)$ is a continuous map. Then (X, τ_1, τ_2) is an almost (i, j)-Baire bispace, where $i, j = 1, 2,\ i \neq j$.*

Proof. Assume that in the bispace (X, τ_1, τ_2) there is a set $U \in \tau_i \setminus \{\emptyset\}$ such that $U = \bigcup_{n=1}^{\infty} G_n$, where $G_n \in (i, j) - ND(X)$ for each $n \in \mathbb{N}$. Let us consider a family $\{C_n\}_{n \in \mathbb{N}}$, where $C_1 \equiv G_1$ and $C_n \equiv G_n \setminus \bigcup_{k=1}^{n-1} G_k$ for every natural $n \geq 2$. Then we get $U = \bigcup_{n=1}^{\infty} C_n$ and $C_n \in (i, j) - ND(X)$ for each $n \in \mathbb{N}$. Moreover, it can be easily shown that $C_n \cap C_m = \emptyset$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

Next, using the set $W = \{y_n | \{y_n\} \in \gamma_i\}_{n = \overline{0;\infty}}$, we define a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ as follows. Suppose that $f(C_n) = \{y_n\}$ for all $n \geq 1$ and $f(X \setminus U) = \{y_0\}$. The conditions of this theorem imply the existence of a set $A \in D_i(X)$ such that the restriction of the map f on the set A is i-continuous. Note that since $A \in D_i(X)$, we have $U \cap A \neq \emptyset$ and therefore, for any point $x \in U \cap A$ there exists a number $n \in \mathbb{N}$ such that $f(x) = y_n$. Moreover, from the continuity of $f|_A$ we conclude that there exists a neighborhood $V \in \Sigma_i^X(x)$ such that $f(V \cap A) \subseteq \{y_n\}$, i.e. $f(V \cap A) = \{y_n\}$. It is obvious that $f(V \cap U \cap A) = \{y_n\}$ and $V \cap U \cap A \subseteq f^{-1}(\{y_n\}) = C_n$. From the latter implication it follows that $\tau_j \mathrm{cl} C_n \supseteq V \cap U \cap A \neq \emptyset$. By the condition $A \in (i,j) - SO(X)$ we conclude that a nonempty set $T \equiv A \cap V \cap U \in (i,j) - SO(X)$. Therefore the implication $\emptyset \neq \tau_i \mathrm{int} T \subset T \subseteq \tau_j \mathrm{cl} C_n$ yields that $C_n \notin (i,j) - ND(X)$, a contradiction. □

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