SQUEEZING POLYNOMIAL ROOTS A NONUNIFORM DISTANCE

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Abstract. Given a polynomial with all real roots, the Polynomial Root Squeezing Theorem states that moving two roots an equal distance toward each other, without passing other roots, will cause each critical point to move toward $(r_i + r_j)/2$, or remain fixed. In this note, we extend the Polynomial Root Squeezing Theorem to the case where two roots are squeezed together a nonuniform distance.

1. Introduction

Given a polynomial $p(x)$, all of whose roots are real, the Polynomial Root Dragging Theorem [1, 4] states that moving one or more roots of the polynomial to the right will cause every critical point to move to the right, or stay fixed. Moreover, no critical point moves as far as the root that is moved the farthest. But what happens to the position of a critical point when some of the roots are dragged in opposing directions?

The Polynomial Root Squeezing Theorem [2, 3] begins the analysis of this problem. Let $r_1 \leq \cdots \leq r_n$ be the n real roots of $p(x)$ with $r_i \neq r_j$ interior roots. We say that a critical point is *stubborn* if it is a repeated root of $\frac{p(x)}{(x-r_i)(x-r_j)}$, and *ordinary* otherwise. Then the assertion of the Polynomial Root Squeezing Theorem is that if r_i and r_j move equal distances toward each other, without passing other roots, then each stubborn critical point which is not located at r_i or r_j will stay fixed, and each ordinary critical point moves toward $(r_i+r_j)/2$. If r_i or r_j is a repeated root of multiplicity greater than two, one of the repeated critical points will move toward $(r_i + r_j)/2$, while the others will remain fixed. In this case, the moving root which is closest to a given critical point has the most pull on that critical point. Unfortunately, this intuition does not allow us to see what happens when two distinct roots are squeezed together a nonuniform distance.

Throughout the paper we will let $p(x)$ be a polynomial of degree n with n real roots $r_1 \leq r_2 \leq \cdots \leq r_n$ and critical points $c_1 \leq c_2 \leq \cdots \leq c_{n-1}$. Consider two distinct roots $r_i < r_j$ and c_k any ordinary critical point. If we drag r_i to the right, the Polynomial Root Dragging Theorem tells us that c_k will also move to the right. If we then drag r_j to the left, the critical point c_k moves back to the left. But how far must we drag r_j to the left in order for c_k to return to its original position? We call this distance the *threshold value*. For moving r_j back any smaller distance will leave c_k to the right of its original position, and moving r_j any larger distance leaves c_k to the left of its original position. In what follows we present a formula for the threshold value.

2. Squeezing Roots a Nonuniform Distance

The threshold value determines exactly what happens to the position of c_k when two roots are squeezed together a nonuniform distance.

Theorem 2.1 (Threshold Value). *Let $p(x)$ be a monic polynomial of degree n with $r_i < r_j$, $d \le r_{i+1} - r_i$, and $0 < h < d$. Let c_k be any ordinary critical point,*

$$\tilde{p}(x) = (x - r_i - h)(x - r_j + h + \alpha_k)q(x)$$

with

$$q(x) = \prod_{k \neq i,j} (x - r_k)$$

and

$$\alpha_k = \frac{-h(r_j - (r_i + h))q'(c_k)}{q(c_k) + (c_k - (r_i + h))q'(c_k)}. \tag{2.1}$$

Then $\tilde{p}'(c_k) = 0$.

We let $d \le \min\{r_{i+1} - r_i, r_j - r_i\}$, so that we study the case where two roots are squeezed together without passing other roots. However, when finding the threshold value, r_j may have to pass other roots. In fact, it may even pass through $r_i + h$. Let's consider such an example. If $p(x) = x(x-1)(x-4)$ and we drag $r_2 = 1$ to the right $\frac{1}{2}$ units, the threshold value for c_1, when we drag $r_3 = 4$ to the left, is 2.691569405. That is, $p(x)$ and $\tilde{p}(x) = x(x - 1.5)(x - 1.308430595)$ have the same first critical point where we have moved $r_2 = 1$ to $r_2 + h = 1.5$ and $r_3 = 4$ to $r_3 - (h + \alpha_k) = 1.308430595$.

The hypothesis that c_k is an ordinary critical point does not need to be weakened to include stubborn critical points. If c_k is a stubborn critical point it does not move in response to r_i and r_j being squeezed. In this case $q(c_k) = q'(c_k) = 0$ and equation (2.1) is undefined, as it should be, since any value of α_k will leave the critical point c_k fixed. However, when c_k is an ordinary critical point the value of α_k is unique.

Proof. Let $r_i < r_j$ be two distinct roots of p and c_k any ordinary critical point. Since

$$p'(x) = (x - r_i + x - r_j)q(x) + (x - r_i)(x - r_j)q'(x)$$

and

$$\begin{aligned}\tilde{p}'(x) &= (x - r_i + x - r_j + \alpha_k)q(x) \\ &\quad + (x^2 - (r_i + r_j)x + r_ir_j + h(r_j - r_i - h) - \alpha_k(h + r_i - x))q'(x),\end{aligned}$$

it follows that

$$\tilde{p}'(x) - p'(x) = \alpha_k q(x) + (h(r_j - r_i - h) - \alpha_k(h + r_i - x))q'(x).$$

As c_k is a critical point of $p(x)$,

$$\tilde{p}'(c_k) = \alpha_k q(c_k) + (h(r_j - (r_i + h)) - \alpha_k(h + r_i - c_k))q'(c_k).$$

Setting $\tilde{p}'(c_k) = 0$ yields

$$(q(c_k) + (c_k - (r_i + h))q'(c_k))\,\alpha_k = -h(r_j - (r_i + h))q'(c_k). \tag{2.2}$$

In order to solve for α_k, we must show that

$$q(c_k) + (c_k - (r_i + h))q'(c_k)$$

is not zero for h satisfying $0 < h < r_j - r_i$. By shifting $p(x)$ we can assume that $c_k = 0$, so we need to show that

$$q(0) - (r_i + h)q'(0) \tag{2.3}$$

is not zero for $0 < h < r_j - r_i$. If $q'(0) = 0$, then (since c_k is an ordinary critical point) $q(0) \neq 0$ and (2.3) is not zero. Therefore, we can assume that $q'(0) \neq 0$ in what follows.

By differentiating $p(x) = (x - r_i)(x - r_j)q(x)$ at $x = 0$, we get

$$(r_i + r_j)q(0) = r_ir_jq'(0). \tag{2.4}$$

Multiplying (2.3) by $r_i + r_j$ and substituting the value of $(r_i + r_j)q(0)$ from the last equation, it suffices to show

$$r_ir_jq'(0) - (r_i + r_j)(r_i + h)q'(0) \neq 0.$$

Since $q'(0) \neq 0$, we can factor out $-q'(0)$ and show that

$$-r_ir_j + (r_i + r_j)(r_i + h) = r_i^2 + h(r_i + r_j)$$

is not zero in the range $0 < h < r_j - r_i$. At $h = 0$ the expression's value is r_i^2 and at $h = r_j - r_i$ its value is r_j^2. Because $r_i < r_j$, at least one of r_i^2 and r_j^2 is positive, and the other is non-negative. Because the expression is a linear function of h, $r_i^2 + h(r_i + r_j)$ and hence $q(c_k) + (c_k - (r_i + h))q'(c_k)$ is non-zero throughout the range $0 < h < r_j - r_i$. Therefore,

$$\alpha_k = \frac{-h(r_j - (r_i + h))q'(c_k)}{q(c_k) + (c_k - (r_i + h))q'(c_k)}$$

is defined, and for this value of α_k, $\tilde{p}'(c_k) = 0$.

□

The cases where $r_i = c_k$ or $r_j = c_k$ are trivial. As intuition suggests, if $r_i = c_k$, then

$$h + \alpha_k = h + \frac{-h(r_j - (r_i + h))q'(c_k)}{(r_i - (r_i + h))q'(c_k)} = h + \frac{-h(r_j - (r_i + h))}{(-h)} = r_j - r_i.$$

Likewise, if $r_j = c_k$, then $h + \alpha_k = 0$.

The Polynomial Root Dragging Theorem suggests that $0 \leq h + \alpha_k \leq r_j - r_i$. This is in fact true.

Lemma 2.2. *Under the hypothesis of Theorem 2.1,*

$$h + \alpha_k = \frac{hr_j^2}{r_i^2 + h(r_i + r_j)}$$

with $0 \leq h + \alpha_k \leq r_j - r_i$.

Proof. By shifting $p(x)$, we can assume that $c_k = 0$.
(More generally, one can show that

$$h + \alpha_k = \frac{h(r_j - c_k)^2}{(r_i - c_k)^2 + h(r_i + r_j - 2c_k)}.$$

However, the starting point for this formula is to consider an ordinary critical point c_k. So for simplicity we shift the polynomial making $c_k = 0$.) Therefore,

$$\begin{aligned} h + \alpha_k &= h + \frac{-h(r_j - (r_i + h))q'(0)}{q(0) - (r_i + h)q'(0)} \\ &= \frac{hq(0) - hr_j q'(0)}{q(0) - (r_i + h)q'(0)}. \end{aligned}$$

If $r_i r_j \neq 0$, equation (2.4) implies that

$$\begin{aligned} h + \alpha_k &= \frac{hq(0) - hr_j \frac{(r_i + r_j)}{r_i r_j} q(0)}{q(0) - (r_i + h)\frac{(r_i + r_j)}{r_i r_j} q(0)} \\ &= \frac{hr_j^2}{r_i^2 + h(r_i + r_j)}. \end{aligned}$$

If $r_i r_j = 0$, then either $r_i = 0$ or $r_j = 0$. If $r_i = 0$ (since $r_i \neq r_j$, $r_j \neq 0$), Equation (2.4) implies that $q(0) = 0$. In this case

$$\begin{aligned} h + \alpha_k &= \frac{hr_j q'(0)}{hq'(0)} \\ &= r_j. \end{aligned}$$

If $r_j = 0$, similar work shows that $h + \alpha_k = 0$. So when $r_i r_j = 0$, the formula

$$h + \alpha_k = \frac{hr_j^2}{r_i^2 + h(r_i + r_j)}$$

still holds.

Since,

$$\frac{d}{dh}\left(\frac{hr_j^2}{r_i^2 + h(r_i + r_j)}\right) = \frac{r_i^2 r_j^2}{(r_i^2 + h(r_i + r_j))^2} \geq 0$$

and $0 < h < r_j - r_i$, we have that

$$0 \leq h + \alpha_k \leq \frac{(r_j - r_i)r_j^2}{r_i^2 + (r_j - r_i)(r_i + r_j)} = r_j - r_i.$$

□

We now are ready to show that c_k is the kth critical point of $\tilde{p}(x)$.

Theorem 2.3. *Under the hypothesis of Theorem 2.1, denote* $\tilde{r}_i = r_i + h$, $\tilde{r}_j = r_j - h - \alpha_k$ *and the critical points of* $\tilde{p}(x)$ *by* $\tilde{c}_1 \leq \tilde{c}_2 \leq \cdots \leq \tilde{c}_{n-1}$. *Then* $c_k = \tilde{c}_k$.

Proof. We will show that c_k is the kth critical point of $\tilde{p}(x)$. Since $0 \leq h + \alpha_k \leq r_j - r_i$, it follows that if $c_k \leq r_i$ or $r_j \leq c_k$, then neither root crosses c_k. Therefore c_k will be the kth critical point of $\tilde{p}(x)$.

When $r_i < c_k < r_j$, we show that there are exactly three possibilities:

$$\tilde{r}_i < c_k < \tilde{r}_j,$$

$$\tilde{r}_j < c_k < \tilde{r}_i,$$

$$\tilde{r}_i = c_k = \tilde{r}_j.$$

By shifting $p(x)$, we can assume $c_k = 0$, so that $r_i < 0 < r_j$. We first show that if $h < |r_i| = -r_i$, then $h + \alpha_k < r_j$. The proof of Lemma 2.2 implies that $h + \alpha_k = \dfrac{hr_j^2}{r_i^2 + h(r_i + r_j)}$ is a nondecreasing function of h. Since $0 < h < -r_i$,

$$h + \alpha_k < \frac{-r_i r_j^2}{r_i^2 - r_i(r_i + r_j)} = r_j.$$

Therefore, if $\tilde{r}_i < c_k$, then $c_k < \tilde{r}_j$. A similar argument shows that if $\tilde{r}_i > c_k$, then $c_k < \tilde{r}_j$.

We now show that if $h = -r_i$, then $h + \alpha_k = r_j$. In this case,

$$\begin{aligned} h + \alpha_k &= h + \frac{-h(r_j - (r_i + h))q'(0)}{q(0) - (r_i + h)q'(0)} \\ &= -r_i + \frac{r_i r_j q'(0)}{q(0)} \\ &= \frac{-r_i q(0) + (r_i + r_j)q(0)}{q(0)} \\ &= r_j. \end{aligned}$$

Therefore, if $\tilde{r}_i = c_k$, then $c_k = \tilde{r}_j$.

When $r_i < c_k < r_j$, it is now easy to see that c_k will be the kth critical point of $\tilde{p}(x)$, we simply count the number of roots to the left of c_k. However, in each case, since r_i and r_j are the only moved roots, it is clear that c_k is the kth critical point of $\tilde{p}(x)$. □

In general, it seems unrealistic to know what happens to a given critical point, when squeezing roots a nonuniform distance, without accounting for the distances between the critical point and each moving root which is done in Lemma 2.2. It remains an open question to find similar estimates when more than two roots are moved in opposing directions (a uniform or nonuniform distance). This could prompt some interesting undergraduate research.

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