FRACTION SETS FOR BASIC DIGIT SETS

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Abstract. A finite set of integers D with $0 \in D$ is *basic* for the base $b \in \mathbb{Z}$ if every $z \in \mathbb{Z}$ can be written uniquely as an integer in the base b with digits from D. Since such a base representation is unsigned, basic sets must have a negative base b or some negative integers in D. The *fraction set* for (b, D) is the set of all representable numbers with integer part zero. We show that if (b, D) is basic and D consists of consecutive digits, then the fraction set is an interval of length one whose endpoints have redundant representations. Furthermore, we show that if D does not consist of consecutive integers, then F is disconnected.

1. Basic Digit Sets

Let $b \in \mathbb{Z}$ and $D \subset \mathbb{Z}$. A representation of a number z in the *base* b with *digit set* D is

$$z = \sum_{k=-\infty}^{n} a_k b^k$$

for some $n \in \mathbb{Z}$ and $a_k \in D$. We denote $\sum_{k=-\infty}^{-1} a_k b^k$ as the *fractional part of* z and $\sum_{k=0}^{n} a_k b^k$ as the *integer part of* z. Furthermore, we denote the base and digit set pair as (b, D).

Definition 1. *Let $b \in \mathbb{Z}$, $|b| \geq 2$, D a finite subset of $\mathbb{Z}$, and $0 \in D$. Then (b, D) is basic if every integer z has a unique representation of the form*

$$z = \sum_{k=0}^{n} a_k b^k$$

for some $n \in \mathbb{N}$ and $a_k \in D$. Note that such a representation has no fractional part.

The usual base and digit pair sets (b, D), with $b > 1$ and $D = \{0, 1, 2, \ldots, b-1\}$ are not basic since negative integers have no such representation. Thus we see that basic sets must have a negative base b or some negative integers in D. Matula has characterized the basic sets with the following theorem.

Theorem 1. [1, Theorem 5] *(b, D) is basic if and only if*
i) D is a complete residue system modulo b (in particular, $|D| = |b|$) and
ii) For each $n \in \mathbb{N}$, the set of all n-digit numbers with digits in D contains no non-zero multiples of $b^n - 1$.

Matula has also answered the question of the existence of representations for all real numbers in a basic set.

Theorem 2. [1, Theorem 10] *If (b, D) is basic then every real number has a representation.*

2. Fraction Sets

For a given base (b, D), denote by $F = F(b, D)$ the set of all fractions (i.e. the set of all representable numbers with zero integer part). Let (b, D) be basic. We will show that if D consists of consecutive digits, then F is an interval of length one whose endpoints have redundant representations. Furthermore, we show that if D does not consist of consecutive integers, then F is disconnected.

Lemma 1. *Let $b \in \mathbb{Z}$ with $|b| \geq 2$ and let $D \subset \mathbb{Z}$. Let $a = min(D)$ and $A = max(D)$.*

(1) *If $b > 0$, then*
$$min(F) = \frac{a}{b-1} \quad and \quad max(F) = \frac{A}{b-1}.$$
(2) *If $b < 0$, then*
$$min(F) = \frac{Ab + a}{b^2 - 1} \quad and \quad max(F) = \frac{ab + A}{b^2 - 1}.$$
(3) *For all b,*
$$max(F) - min(F) = \frac{A - a}{|b| - 1}.$$

Proof.

(1) Let $b > 0$. Then for $\sum_{i=-\infty}^{-1} a_i b^i \in F$, each term $a_i b^i$ in the sum has minimum value when $a_i = a$ and maximum value when $a_i = A$. Thus, the minimum element of F is
$$\sum_{i=-\infty}^{-1} ab^i = .\overline{a} = a \cdot \sum_{i=-\infty}^{-1} b^i = a \cdot \frac{\frac{1}{b}}{1 - \frac{1}{b}} = \frac{a}{b-1}.$$
Similarly, the maximum element of F is
$$\sum_{i=-\infty}^{-1} Ab^i = .\overline{A} = A \cdot \sum_{i=-\infty}^{-1} b^i = A \cdot \frac{\frac{1}{b}}{1 - \frac{1}{b}} = \frac{A}{b-1}.$$

(2) Let $b < 0$. Then the minimum and maximum values of the terms $a_i b^i$ depend on the parity of i.

(a) If i is even, then $b^i > 0$ so that the minimum values occur when $a_i = a$ and the maximum values occur when $a_i = A$. Thus, the even powered terms of the minimum element of F are

$$ab^{-2} + ab^{-4} + \cdots .$$

Similarly, the even powered terms of the maximum element of F are

$$Ab^{-2} + Ab^{-4} + \cdots .$$

(b) When i is odd, we have that $b^i < 0$ so that the minimum values occur when $a_i = A$ and the maximum values occur when $a_i = a$. Thus, the odd powered terms of the minimum element of F are

$$Ab^{-1} + Ab^{-3} + \cdots .$$

Similarly, the odd powered terms of the maximum element of F are

$$ab^{-1} + ab^{-3} + \cdots .$$

Collecting all such terms, we see that the minimum element of F is of the form

$$.\overline{Aa} = A\left(\frac{1}{b} + \frac{1}{b^3} + \cdots\right) + a\left(\frac{1}{b^2} + \frac{1}{b^4} + \cdots\right)$$
$$= A \cdot \frac{\frac{1}{b}}{1 - \frac{1}{b^2}} + a \cdot \frac{\frac{1}{b^2}}{1 - \frac{1}{b^2}} = \frac{Ab + a}{b^2 - 1}.$$

Similarly, the maximum element of F is of the form

$$.\overline{aA} = a\left(\frac{1}{b} + \frac{1}{b^3} + \cdots\right) + A\left(\frac{1}{b^2} + \frac{1}{b^4} + \cdots\right)$$
$$= a \cdot \frac{\frac{1}{b}}{1 - \frac{1}{b^2}} + A \cdot \frac{\frac{1}{b^2}}{1 - \frac{1}{b^2}} = \frac{ab + A}{b^2 - 1}.$$

(3) (a) Let $b > 0$. Then by part 1 above,

$$max(F) - min(F) = \frac{A}{b-1} - \frac{a}{b-1}$$

and the result follows.

(b) Let $b < 0$. Then by part 2 above,

$$max(F) - min(F) = \frac{ab + A}{b^2 - 1} - \frac{Ab + a}{b^2 - 1} = \frac{(A - a)(1 - b)}{b^2 - 1}$$

and the result follows.

□

Having established the extrema for F, we will denote the interval $I = [min(F), max(F)]$ so that $F \subset I$ and $|I| = \frac{A-a}{|b|-1}$.

Theorem 3. *If (b, D) is a basic set and D consists of consecutive integers, then F is an interval of length 1 and the endpoints of F have redundant representations.*

Proof. Since $D = \{a, \ldots, A\}$ contains $|b|$ consecutive digits, we have that $A - a = |b| - 1$. By part 3 of Lemma 1 we have

$$|I| = max(F) - min(F) = \frac{A-a}{|b|-1} = 1.$$

We now claim that $F = I$. By construction, the endpoints of I are in F. Let x be in the interior of I. Since (b, D) is basic, x has a representation. If x has non-zero integer part n, then $x-n$ has zero integer part and $x-n \in F$.

(1) If $n \geq 1$ then $x - n \leq x - 1 < max(F) - 1 = min(F)$.
(2) If $n \leq -1$ then $x - n \geq x + 1 > min(F) + 1 = max(F)$.

In either case, we have an element $x - n \in F$ that is not in I. This contradicts that $F \subset I$. Thus, every element of I has zero integer part and $I = F$.

Since (b, D) is basic, both -1 and 1 have unique representations of the form $-1 = \sum_{k=0}^{n} c_k b^k$ and $1 = \sum_{k=0}^{m} d_k b^k$ for some $n, m \geq 0$ and $c_k, d_k \in D$. Additionally, the endpoints of F have representations with zero integer part of the form $min(F) = \sum_{k=-\infty}^{-1} e_k b^k$ and $max(F) = \sum_{k=-\infty}^{-1} f_k b^k$ for some $e_k, f_k \in D$. Thus, we have the redundant representations

$$min(F) = \sum_{k=-\infty}^{-1} e_k b^k = max(F) - 1 = \sum_{k=-\infty}^{-1} f_k b^k + \sum_{k=0}^{n} c_k b^k$$

and

$$max(F) = \sum_{k=-\infty}^{-1} f_k b^k = min(F) + 1 = \sum_{k=-\infty}^{-1} e_k b^k + \sum_{k=0}^{m} d_k b^k.$$

□

Example 1. *Both balanced ternary* $(3, \{-1, 0, 1\})$ *and negabinary* $(-2, \{0, 1\})$ *are basic* [1, p. 1137]. *We use Lemma 1 and Theorem 3 to compute their fraction sets.*

(1) *For balanced ternary, we have that* $min(F) = -\frac{1}{2}$ *and* $max(F) = \frac{1}{2}$ *so that* $F = [-\frac{1}{2}, \frac{1}{2}]$*. The endpoints of* F *have redundant representations of the form*

$$\frac{1}{2} = .\overline{1} = 1.\overline{-1}$$

$$-\frac{1}{2} = .\overline{-1} = -1.\overline{1}.$$

(2) *For negabinary, we have that* $min(F) = -\frac{2}{3}$ *and* $max(F) = \frac{1}{3}$ *so that* $F = [-\frac{2}{3}, \frac{1}{3}]$*. The endpoints of* F *have redundant representations of the form*

$$\frac{1}{3} = .\overline{01} = 1.\overline{10}$$

$$-\frac{2}{3} = .\overline{10} = 11.\overline{01}.$$

Theorem 4. *If* (b, D) *is basic and* D *does not consist of consecutive digits, then* F *is disconnected.*

Proof. Since D does not consist of consecutive digits, $A - a > |b| - 1$ and thus $|I| > 1$. Let $A_{|b|} = \{i \cdot b^j : i, j \in \mathbb{Z}\}$. Then $A_{|b|}$ is dense in $\mathbb{R}$ and each $x \in A_{|b|}$ has a unique representation in (b, D) [1, Lemma 14].

Suppose $|I| = 1 + \epsilon$ for some $\epsilon > 0$. Choose $x \in A_{|b|}$ so that $0 < x - min(F) < \frac{\epsilon}{2}$ and thus $x \in I$. We then have that

$$1 + x - min(F) < 1 + \frac{\epsilon}{2} < |I|.$$

Thus $1 + x < max(F)$ and $1 + x \in I$. If $x \notin F$, then F is disconnected. Suppose $x \in F$. Then x has a representation of the form $x = \sum_{k=-\infty}^{-1} a_k b^k$ with $a_k \in D$. Since (b, D) is basic, 1 has a representation of the form $1 = \sum_{j=0}^{n} c_j b^j$ for some $n \geq 0$ and $c_j \in D$. Thus $1 + x$ has a representation $1 + x = \sum_{j=0}^{n} c_j b^j + \sum_{k=-\infty}^{-1} a_k b^k$ with non-zero integer part.

Note that $A_{|b|}$ is the set of real numbers that have a terminating base $|b|$ expansion with digits in $\{0, 1, \ldots, |b| - 1\}$. Therefore, since $x \in A_{|b|}$, so is $1 + x$. Thus $1 + x$ must have a unique representation in (b, D), and that representation must be the one given above with non-zero integer part. Thus $1 + x \in I \setminus F$, and F is disconnected. □

Example 2. *The base and digit set* $(3, \{-7, 0, 1\})$ *is basic* [1, Theorem 8]*. Using Lemma 1, we have that* $min(F) = -\frac{7}{2}$ *and* $max(F) = \frac{1}{2}$*. By Theorem 4,* F *is a proper disconnected subset of* $I = [-\frac{7}{2}, \frac{1}{2}]$*.*

References

[1] D. W. Matula, *Basic digit sets for radix representation*, Journal of the Association for Computing Machinery, **29.4** (1982), 1131–1143.

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