ON SOME THEOREMS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. In this note, we define adherent and accumulation points of a set in an intuitionistic fuzzy metric space, and prove the Sierpinski and Hine Theorem for intuitionistic fuzzy metric spaces.

1. Introduction

Zadeh [11] introduced the concept of fuzzy sets. Atanassov [1] generalized fuzzy sets by introducing and studying intuitionistic fuzzy sets. Recently, using the idea of intuitionistic fuzzy sets, Park [10] introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [3]. In a series of papers the authors [2–7,10] are systematically developing the theory of fuzzy structure on metric spaces.

In this note, we define the notions of adherent and accumulation points, and prove that in an intuitionistic fuzzy metric space, each compact subset is a complete set. We also prove a new version of the Hine Theorem and the Sierpinsky Theorem for intuitionistic fuzzy metric spaces. Our results also extend, generalize, and make fuzzy several results on metric spaces and fuzzy metric spaces.

2. Preliminaries

Definition 1. *A binary operation* $*\colon [0,1]\times[0,1]\to[0,1]$ *is a continuous t-norm if* $*$ *satisfies the following conditions:*

(a) $*$ *is commutative and associative;*
(b) $*$ *is continuous;*
(c) $a*1=a$ *for all* $a\in[0,1]$*;*
(d) $a*b\le c*d$ *whenever* $a\le c$ *and* $b\le d$*, and* $a,b,c,d\in[0,1]$.

Definition 2. *A binary operation* $\diamond\colon [0,1]\times[0,1]\to[0,1]$ *is a continuous t-conorm if* $\diamond$ *satisfies the following conditions:*

(a) $\diamond$ *is commutative and associative;*
(b) $\diamond$ *is continuous;*
(c) $a\diamond 0 = a$ *for all* $a\in[0,1]$*;*
(d) $a\diamond b\le c\diamond d$ *whenever* $a\le c$ *and* $b\le d$*, and* $a,b,c,d\in[0,1]$*.*

Remark 3. *The concepts of triangular norms (shortly t-norms) and triangular conorms (shortly t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and fuzzy unions, respectively. These concepts were originally introduced by Menger [9]. Several examples for these concepts were proposed by many authors [2-7,9,10].*

Remark 4. *If* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm and* $r_i\in(0,1)$*,* $1\le i\le 7$*, then*

(a) If $r_1>r_2$*, there are* $r_3,r_4\in(0,1)$ *such that* $r_1*r_3\ge r_2$ *and* $r_2\diamond r_4\le r_1$*.*
(b) If $r_5\in(0,1)$*, there are* $r_6,r_7\in(0,1)$ *such that* $r_6*r_6\ge r_5$ *and* $r_7\diamond r_7\le r_5$*.*

Definition 5 ([10])**.** *A 5-tuple* $(X,M,N,*,\diamond)$ *is said to be an intuitionistic fuzzy metric space (shortly IFM-space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm and* M*,* N *are fuzzy sets on* $X^2\times(0,\infty)$ *satisfying the following conditions: for all* x*,* y*,* $z\in X$*,* s*,* $t>0$*,*

(IFM-1) $M(x,y,t)+N(x,y,t)\le 1$*;*
(IFM-2) $M(x,y,t)>0$*;*
(IFM-3) $M(x,y,t)=1$ *if and only if* $x=y$*;*
(IFM-4) $M(x,y,t)=M(y,x,t)$*;*
(IFM-5) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$*;*
(IFM-6) $M(x,y,.)\colon(0,\infty)\to(0,1]$ *is continuous;*
(IFM-7) $N(x,y,t)>0$*;*
(IFM-8) $N(x,y,t)=N(y,x,t)$*;*
(IFM-9) $N(x,y,t)\diamond N(y,z,s)\ge N(x,z,t+s)$*;*
(IFM-10) $N(x,y,.)\colon(0,\infty)\to(0,1]$ *is continuous.*

Then (M,N) *is called an intuitionistic fuzzy metric on* X*. The functions* $M(x,y,t)$ *and* $N(x,y,t)$ *denote the degree of nearness and the degree of non-nearness between* x *and* y *with respect to* t*, respectively.*

Remark 6. *Every fuzzy metric space* $(X,M,*)$ *is an IFM-space of the form* $(X,M,1-M,*,\diamond)$ *such that t-norm* $*$ *and t-conorm* $\diamond$ *are associated, i.e.* $x\diamond y=1-((1-x)*(1-y))$ *for all* $x,y\in[0,1]$*. But the converse is not true.*

Definition 7 ([10]). *Let $(X, M, N, *, \diamond)$ be an IFM-space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r,\ N(x, y, t) < r\}$. Let $\tau_{(M,N)}$ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $r \in (0, 1)$ such that $B(x, r, t) \subset A$. Then $\tau_{(M,N)}$ is a topology on X (induced by the intuitionistic fuzzy metric (M, N)). A sequence $\{x_n\}$ in X converges to x in X if and only if $M(x_n, x, t)$ tends to 1 and $N(x_n, x, t)$ tends to 0 as n tends to ∞, for each $t > 0$. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, \varepsilon) > 1 - \lambda$ and $N(x_n, x_m, \varepsilon) < \lambda$ for all $n, m \geq n_0$. An IFM-space is called complete if every Cauchy sequence is convergent.*

Theorem 8 ([10]). *In an intuitionistic fuzzy metric space every compact set is closed and intuitionistic fuzzy bounded.*

3. Main Results

Definition 9. *Let $(X, M, N, *, \diamond)$ be an IFM-space and $S \subseteq X$.*

(a) $x \in X$ is said to be adherent to S if

$$B(x, r, t) \cap S \neq \emptyset \text{ for each } r \in (0, 1) \text{ and } t > 0,$$

(b) $x \in X$ is said to be an accumulation point of S if

$$(B(x, r, t) \cap S) \setminus \{x\} \neq \emptyset \text{ for each } r \in (0, 1) \text{ and } t > 0.$$

Theorem 10. *Let $(X, M, N, *, \diamond)$ be an IFM-space and S be a compact subset of X. Then each infinite subset of S has an accumulation point in A.*

Proof. The proof is the same as metric spaces [8]. □

Theorem 11. *Let $(X, M, N, *, \diamond)$ be an IFM-space and S be a compact subset of X. Then S is complete.*

Proof. Let $\{x_n\} \subset S$ be a Cauchy sequence and $A = \{x_1, x_2, \ldots\}$ be the range of $\{x_n\}$. If A is a finite set, then $\{x_n\}$ is convergent to an element of S. If A is an infinite set, then A has an accumulation point x_0 in S. We will show that $x_n \to x_0$.

Fix $t > 0$. For a given $\varepsilon \in (0, 1)$, we can find a δ such that $(1 - \delta) * (1 - \delta) \geq 1 - \varepsilon$ and $\delta \diamond \delta \leq \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$, we have $M(x_n, x_m, t/2) > 1 - \delta$ and $N(x_n, x_m, t/2) < \delta$. Since x_0 is an accumulation point of S, we have $(B(x_0, \delta, t/2) \cap S) \setminus \{x\} \neq \emptyset$. Consequently, there exists an $m \geq n_0$ such that $x_m \in B(x_0, \delta, t/2)$. Now, for each $n \geq n_0$, we have

$$\begin{aligned} M(x_n, x_0, t) &\geq M(x_n, x_m, t/2) * M(x_m, x_0, t/2) \\ &\geq (1 - \delta) * (1 - \delta) \geq 1 - \varepsilon, \end{aligned}$$

$$\begin{aligned} N(x_n, x_0, t) &\leq N(x_n, x_m, t/2) \diamond N(x_m, x_0, t/2) \\ &\leq \delta \diamond \delta \leq \varepsilon. \end{aligned}$$

Since ε was arbitrary, $M(x_n, x_0, t) \to 1$ and $N(x_n, x_0, t) \to 0$, that is $x_n \to x_0$. Hence, S is complete. □

Theorem 12. *Let $(X, M, N, *, \diamond)$ be an IFM-space, S be a compact subset of X and $\{x_n\} \subset S$. Then $\{x_n\}$ has a convergent subsequence in X.*

Proof. Let $A = \{x_n : n \in \mathbb{N}\}$. If A is finite, then there exists $x \in A$ and increasing sequence $\{n_i\}$ such that $x_{n_1} = x_{n_2} = \cdots = x$. Hence, the subsequence $\{x_{n_i}\}$ is convergent to x. If A is infinite, then A has an accumulation point, say x. Fix $t > 0$ and choose n_1 such that $x_{n_1} \in B(x, 1/2, t)$. Assume that points $x_{n_1}, \ldots, x_{n_{i-1}}$ are properly chosen. Since $\big(B(x, 2^{-i}, t) \cap A\big) \setminus \{x\} \neq \emptyset$, we can choose $n_i > n_{i-1}$ such that $x_{n_i} \in B(x, 2^{-i}, t)$. Therefore, $M(x_{n_i}, x, t) > 1 - 2^{-i}$ and $N(x_{n_i}, x, t) < 2^{-i}$, that is $x_n \to x_0$. Hence, $\{x_{n_i}\}$ is convergent. □

Theorem 13 (Intuitionistic Fuzzy Hine Theorem)**.** *Let $(X, M, N, *, \diamond)$ and $(Y, O, P, \star, \triangle)$ be IFM-spaces and $f\colon X \to Y$ be a continuous map. If X is compact, then f is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. Then there are $\varepsilon \in (0, 1)$, $t > 0$, $\{p_n\}$ and $\{q_n\}$ such that $O(f(p_n), f(q_n), t) < 1 - \varepsilon$ and $P(f(p_n), f(q_n), t) > \varepsilon$, but $M(p_n, q_n, t) \to 0$ and $N(p_n, q_n, t) \to 1$. Since $\{p_n\}$ is a subset of the compact set X, there is convergent subsequence $\{p_{n_i}\}$ such that $p_{n_i} \to p$. Consider the inequalities $M(q_{n_i}, p, 2t) \geq M(q_{n_i}, p_{n_i}, t) * M(p_{n_i}, p, t)$ and $N(q_{n_i}, p, 2t) \leq N(q_{n_i}, p_{n_i}, t) \diamond N(p_{n_i}, p, t)$, and let $i \to \infty$. Thus, we get $M(q_{n_i}, p, 2t) \to 1$ and $N(q_{n_i}, p, 2t) \to 0$ so $q_{n_i} \to p$. But f is continuous at $p \in X$, hence, $\lim_{i\to\infty} f(p_{n_i}) = f(p)$ and $\lim_{i\to\infty} f(q_{n_i}) = f(p)$. Therefore, $O(f(p_n), f(q_n), t) \geq 1 - \varepsilon$ and $P(f(p_n),\ f(q_n), t) \leq \varepsilon$ for large i, which are contradictions. □

Theorem 14 (Intuitionistic Fuzzy Sierpinski Theorem)**.** *Let $(Y, O, P, \star, \triangle)$ be an IFM-space and X be a complete subset of Y. Then X is a G_δ subset of Y.*

Proof. Let (O, P) be the intuitionistic fuzzy metric on Y and (M, N) be an intuitionistic fuzzy metric on X that induces the same topology on X as (O, P) does. For each $x \in X$ and each $n \in \mathbb{N}$, let $r_n(x)$ be a positive real number such that $r_n(x) < n^{-1}$, $M(w, x, t) > 1 - \frac{1}{n}$ and $N(w, x, t) < \frac{1}{n}$ whenever $w \in X$, and $O(w, x, t) > 1 - r_n(x)$ and $P(w, x, t) < r_n(x)$ for each $t > 0$. Suppose that $B_n(x) = \{B(x, r_n(x), t) : t > 0\}$ and $G_n = \cup\{B_n(x) : x \in X\}$ for each $n \in \mathbb{N}$ and $T = \cap\{G_n : n \in \mathbb{N}\}$. Then T is a G_δ subset of Y which clearly contains X. Now, we will prove that $T \subseteq X$.

Let $x_0 \in T$, then $x_0 \in G_n$ for each n, hence, for each n there exists $x_n \in X$ such that $x_0 \in B_n(x)$. Hence, $O(x_0, x_n, t) > 1 - r_n(x) > 1 - \frac{1}{n}$ and $P(x_0, x_n, t) < r_n(x) < \frac{1}{n}$ for each $t > 0$. Thus, $x_n \to x_0$ in Y. Now, let $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ such that $(1 - 1/n) * (1 - 1/n) > 1 - \varepsilon$ and $(1/n) \diamond (1/n) < \varepsilon$. Let $m \in \mathbb{N}$ such that $(1-1/m) \star O(x_0, x_N, t) > 1 - r_N(x)$ and $(1/m) \triangle P(x_0, x_N, t) < r_N(x)$ for each $t > 0$. Now, for every $k > 0$, we have

$$\begin{aligned} O(x_k, x_N, 2t) &\geq O(x_k, x_0, t) \star O(x_N, x_0, t) \\ &\geq \left(1 - \frac{1}{k}\right) \star O(x_N, x_0, t), \end{aligned}$$

$$\begin{aligned} P(x_k, x_N, 2t) &\leq P(x_k, x_0, t) \triangle P(x_N, x_0, t) \\ &\leq \frac{1}{k} \triangle P(x_N, x_0, t). \end{aligned}$$

If $k > m$ then $O(x_k, x_N, 2t) \geq (1 - \frac{1}{m}) \star O(x_N, x_0, t) > 1 - r_N(x)$ and $P(x_k, x_N, 2t) \leq \frac{1}{m} \triangle P(x_N, x_0, t) < r_N(x)$. Hence, $M(x_k, x_N, 2t) > 1 - \frac{1}{n}$ and $N(x_k, x_N, 2t) < \frac{1}{n}$. If $k, l > m$, then

$$\begin{aligned} M(x_k, x_l, 4t) &\geq M(x_k, x_N, 2t) * M(x_l, x_N, 2t) \\ &> \left(1 - \frac{1}{n}\right) * \left(1 - \frac{1}{n}\right) > 1 - \varepsilon, \end{aligned}$$

$$\begin{aligned} N(x_k, x_l, 4t) &\leq N(x_k, x_N, 2t) \diamond N(x_l, x_N, 2t) \\ &< \frac{1}{n} \diamond \frac{1}{n} < \varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and so is convergent to some member of X. Since $x_n \to x_0$ in Y, it follows that $x_0 \in X$. Hence, $T \subseteq X$. $\square$

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87–96.

[2] M. Amini and R. Saadati, *Topics in fuzzy metric spaces*, J. Fuzzy Math., **11** (2003), 765–768.

[3] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64** (1994), 395–399.

[4] S. Kutukcu, D. Turkoglu, and C. Yildiz, *Common fixed points of compatible maps of type (β) on fuzzy metric spaces*, Commun. Korean Math. Soc., **21** (2006), 89–100.

[5] S. Kutukcu, *A common fixed point theorem for a sequence of self maps in intuitionistic fuzzy metric spaces*, Commun. Korean Math. Soc., **21** (2006), 679–687.

[6] S. Kutukcu, *A fixed point theorem for contraction type mappings in Menger spaces*, American J. Appl. Sci., **4** (2007), 371–373.

[7] S. Kutukcu, D. Turkoglu, and C. Yildiz, *Some fixed point theorems for multivalued mappings in fuzzy Menger spaces*, J. Fuzzy Math., (to appear).

[8] R. E. Megginsion, *An Introduction on Banach Space Theory*, Springer-Verlag, New York, 1998.
[9] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci., **28** (1942), 535–537.
[10] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals, **22** (2004), 1039–1046.
[11] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8** (1965), 338–353.

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