SPIRAL KNOTS

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Abstract. Spiral knots are a generalization of torus knots we define by a certain periodic closed braid representation. For spiral knots with prime power period, we calculate their genus, bound their crossing number, and bound their m-alternating excess.

1. Introduction

Torus knots are a well-understood class of knots. Among other properties, their crossing number, closed braid representations, normalized Alexander polynomial degrees, and genera are all known. In this paper, we define a class of knots that includes torus knots, and attempt to answer questions regarding all these properties for this larger category.

We will use the standard notation for braid words when describing open braids on n strands, where σ_i denotes the ith strand crossing over the $(i+1)$st strand ($i = 1, 2, \ldots, n-1$). The defining relations in the corresponding braid group are the Artin relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-1$. On closed braids, we have the two additional moves of *conjugation*, $w \mapsto \sigma_i^\epsilon w \sigma_i^{-\epsilon}$, and *stabilization*, $w \leftrightarrow w\sigma_n^\epsilon$ (where $\sigma_n^{\pm\epsilon}$ does not occur in w). A well-known result of Alexander is that every knot has a closed braid representation, and a consequence of Markov's Theorem is that the braid words of two closed-braid representations of the same knot can be related by a sequence of the above four moves (together with the usual reduction $\sigma_i^\epsilon\sigma_i^{-\epsilon} \leftrightarrow 1$).

A knot projection is *m-alternating* if its overcrossings and undercrossings alternate in groups of m as one travels around the projection. Every knot admits a 2-alternating projection [2]. The *m-alternating excess* of a knot is the difference between the minimum number of crossings in any m-alternating projection of the knot and its crossing number. A torus knot $T_{p,q}$ admits a braid word of the form $(\sigma_1\sigma_2\ldots\sigma_{p-1})^q$; it can be thought of as a periodic knot with period q and trivial quotient. In this paper, we study knots which admit braid words of the above form but which allow exponents of either 1 or -1 on the σ_i; we call these knots *spiral knots*. In Section 2, we introduce the notion of spiral links and show that, like torus links, they have one component if and only if $\gcd(p,q)=1$. In Section 3, we use the second Murasugi condition on the Alexander polynomial of spiral knots to both calculate their genus and to bound their crossing number in the case that the period of the knot is a prime power. In Section 4, we compute the $p-1$ and $q-1$ alternating excesses of (p,q) torus knots, find necessary and sufficient conditions for a spiral knot projection to be m-alternating, and find an upper bound for the m-alternating excess of certain spiral knots.

2. Spiral Knots

A (p,q) torus knot admits a p-strand braid word of the form w^q where $w=\sigma_1\sigma_2\cdots\sigma_{p-1}$. We begin by exploring what happens if we focus only on the number of strands, the number of crossings, and the periodic nature of the braid word. Given a braid word $w=x_{i_1}x_{i_2}\cdots x_{i_n}$, where each x_{i_l} is some $\sigma_j^{\pm 1}$, we denote its word length by $|w|=n$. For any integers $n\geq 2$ and $k\geq 1$, an *S-link of type (n,k)* is a link that admits an n-strand braid word of the form w^k with $|w|=n-1$. For example, the braid representations of two S-links of type $(5,2)$ are shown in Figures 1 and 2.

We will call an S-link with one component an *S-knot*. We now show that every S-knot of type (n,k) must have $\gcd(n,k)=1$ and an underlying S-knot of type $(n,1)$ that is trivial. This will be crucial in our later investigations of the Alexander polynomials of S-knots.

Theorem 1. *An S-link of type $(n,1)$ has one component if and only if it is the unknot.*

Proof. Let L_0 be an S-link of type $(n,1)$, let w be its corresponding braid word, and let μ be the permutation associated to w. If L_0 has one component, then μ is an n-cycle. In this case, each σ_i or its inverse must occur exactly once among the $n-1$ letters of w, for if not, then μ would decompose into disjoint cycles, each of which would correspond to a distinct component of L_0. Now repeated stabilization reduces w to the identity. Conversely, if L_0 is the unknot, then it has one component. □

Theorem 2. *An S-link of type (n,k) has one component if and only if the corresponding S-link of type $(n,1)$ has one component and* $\gcd(n,k)=1$.

Proof. Assume L is an S-link of type (n,k) and let L_0 denote the corresponding S-link of type $(n,1)$. Notice L is a periodic knot with period k, quotient L_0, and linking number n. If L_0 is a knot, then L has $\gcd(n,k)$ components by [3]. Thus, if L_0 has one component and $\gcd(n,k)=1$, then L has one component. Conversely, if L has one component, then L_0 must also, and therefore $\gcd(n,k)$ is equal to the number of components of L, i.e. $\gcd(n,k)=1$. □

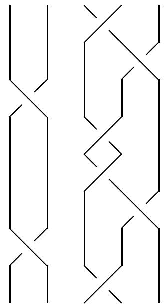

FIGURE 1. S-link $(\sigma_3^{-1}\sigma_4\sigma_1\sigma_3^{-1})^2$

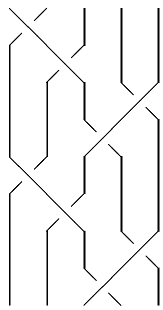

FIGURE 2. S-link $(\sigma_1\sigma_2\sigma_4^{-1}\sigma_3^{-1})^2$

It happens that every S-knot admits a braid word similar to the standard form for torus knots.

Theorem 3. *Let $n \geq 2$ and $k \geq 1$. Every S-knot of type (n,k) admits a braid word of the form $(\sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\ldots\sigma_{n-1}^{\epsilon_{n-1}})^k$, where each $\epsilon_i = \pm 1$. If $\epsilon = (\epsilon_1, \epsilon_2, \ldots \epsilon_{n-1})$, we will call this the* ***spiral knot*** *$S(n,k,\epsilon)$.*

Proof. Let K be an S-knot of type (n,k), let K_0 be the corresponding unknot of type $(n,1)$, and let w^k and w, respectively, be their corresponding braid words. For each i, let $x_i = \sigma_i^{\pm 1}$. By the proof of Theorem 1, each x_i occurs exactly once in w for $i = 1, 2, \ldots, n-1$. By conjugating by the appropriate prefix of w, we can assume that w begins with x_1.

Let $1 \leq r \leq n-3$. If u and v are any words in $x_{r+2}, x_{r+3}, \ldots, x_{n-1}$, we assert that the word $(x_1x_2\cdots x_r u x_{r+1} v)^k$ is Markov equivalent to the word $(x_1x_2\cdots x_r x_{r+1} vu)^k$. To see this, notice that applying commuting Artin relations to $(x_1x_2\cdots x_rux_{r+1}v)^k$ yields $(ux_1x_2\cdots x_rx_{r+1}v)^k$, and then conjugating by u yields $(x_1x_2\cdots x_rx_{r+1}vu)^k$. Repeated application of this result yields the braid word $(x_1x_2\cdots x_{n-1})^k$ for K, as desired. □

Note that we have shown that *any* knot K that admits an n-strand braid word of the form w^k with $|w| = n-1$ is a spiral knot. Spiral knots can be thought of as a generalization of torus knots, where the exponent of each σ_i is allowed to take on one of the values ±1. In particular, the torus knot $T_{p,q}$ is the spiral knot $S(p,q,\epsilon)$ where $\epsilon = (1,1,\ldots,1)$. Three examples of spiral knots are shown in Figure 3.

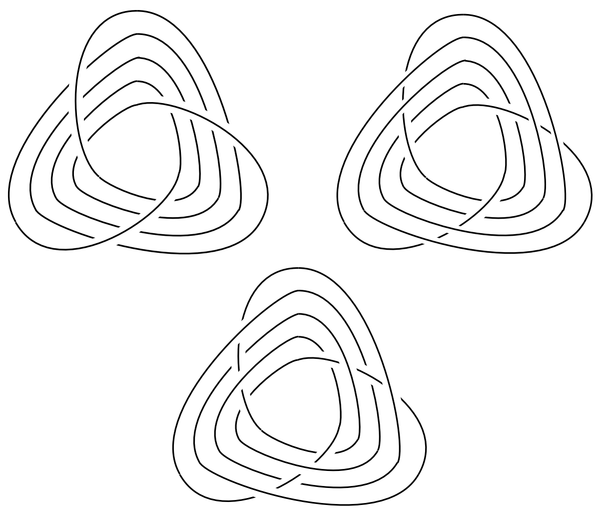

FIGURE 3. Three spiral knots of the form $S(5,3,\epsilon)$, for $\epsilon = (1,1,1,1)$, $(1,1,-,1-,1)$, and $(1,-1,1,-1)$.

The crossing number of a spiral knot $S(n,k,\epsilon)$ is not necessarily equal to $\min((n-1)k,(k-1)n)$ as it is for torus knots. In particular, the standard projection of a spiral knot need not necessarily exhibit the minimal crossing number of the knot. For example, the spiral knot $S(5,2,(1,1,1,1)) = T_{5,2}$ has crossing number $\min(4\cdot 2, 1\cdot 5) = 5$. But $c(S(5,2,(1,1,-1,-1))) = 6$, since $S(5,2,(1,1,-1,-1))$ has braid word $(\sigma_1\sigma_2{\sigma_3}^{-1}{\sigma_4}^{-1})^2$, which is Markov equivalent to a braid word $(\sigma_1{\sigma_2}^{-1}\sigma_1\sigma_1{\sigma_2}^{-1}{\sigma_2}^{-1})$ for 6_3.

3. The Genus of Spiral Knots

In this paper, we use the convention that Alexander polynomials are normalized, that is, have only non-negative powers and positive constant term. Since a spiral knot $S(n,k,\epsilon)$ has period k and quotient $S(n,1,\epsilon)$ with linking number n, a direct application of the second Murasugi condition gives us the following theorem.

Theorem 4. *(Murasugi [4]) If $k = p^r$ for some prime p, then the Alexander polynomial $A(S(n,k,\epsilon)$ satisfies the following congruence.*

$$A(S(n,k,\epsilon)) \equiv A(S(n,1,\epsilon))^k (1+t+\cdots+t^{n-1})^{k-1} \pmod{p}. \tag{1}$$

This fact will allow us to determine the degree of the Alexander polynomial and the genus of spiral knots $S(n,k,\epsilon)$ when k is a prime power. We first obtain some bounds in the general case.

Theorem 5. *The genus g of a spiral knot $S(n,k,\epsilon)$ is bounded as follows.*

$$\frac{\deg(A(S(n,k,\epsilon))}{2} \leq g \leq \frac{(n-1)(k-1)}{2}. \tag{2}$$

Proof. Let $K = S(n,k,\epsilon)$. By [3], we have $\frac{deg(A(K))}{2} \leq g$. According to [1], the genus of a knot projection is $\frac{(c-s+1)}{2}$, where c is the number of crossings and s is the number of Seifert circles for that projection. The standard braid representation of K has n strands, and therefore n Seifert circles. Since this same projection of K has $k(n-1)$ crossings, substitution yields the desired upper bound. $\square$

It is well-known that a torus knot $T_{p,q}$ has genus $\frac{(p-1)(q-1)}{2}$, and that the degree of its normalized Alexander polynomial is $(p-1)(q-1)$. The following theorem extends these results to spiral knots $S(n,k,\epsilon)$ in the case where k is a prime power.

Theorem 6. *If $k = p^r$ for some prime p, then* $\deg(A(S(n,k,\epsilon))) = (n-1)(k-1)$*, and the genus of $S(n,k,\epsilon)$ is $g = \frac{(n-1)(k-1)}{2}$.*

Proof. Let $K = S(n,k,\epsilon)$. By Theorems 1 and 2, $S(n,1,\epsilon)$ is trivial. Thus Theorem 4 implies $A(K) \equiv (1+t+\cdots+t^{n-1})^{k-1} \pmod{p}$. If we let $\deg_p(A(K))$ denote the degree of $A(K)$ when its coefficients are reduced modulo p, then we have $\deg(A(K)) \geq \deg_p(A(K))$. Since $(1+t+\cdots+t^{n-1})^{k-1}$ is monic, $\deg((1+t+\cdots+t^{n-1})^{k-1}) = \deg_p((1+t+\cdots+t^{n-1})^{k-1}) = (n-1)(k-1)$. Thus $\deg(A(K)) \geq (n-1)(k-1)$, and the result follows from Theorem 5. $\square$

One immediate corollary to Theorem 6 is that all spiral knots $S(n,k,\epsilon)$ are nontrivial for k a prime power. We also have the following corollary, which gives bounds on the crossing numbers of spiral knots with prime power periods.

Corollary 1. *If $S(n,k,\epsilon)$ is a spiral knot with $k = p^r$ for some prime p, then $(n-1)(k-1) < c(S(n,k,\epsilon)) \leq (n-1)k$.*

Proof. By [6] we know that $\deg(A(S(n,k,\epsilon))) < c(S(n,k,\epsilon))$. By Theorem 6, $\deg(A(S(n,k,\epsilon))) = (n-1)(k-1)$. But since the standard closed braid projection of $S(n,k,\epsilon)$ has $(n-1)k$ crossings, we have $c(S(n,k,\epsilon)) \leq (n-1)k$. □

4. m-alternating properties of torus knots and spiral knots

The notion of an m-alternating projection is an obvious generalization of the notion of an alternating projection [2]. A projection $P(K)$ of a knot K is *m-alternating* provided at least one of its projection words is of the form $w = (0^m 1^m)^k$, for some $m \geq 0$ and $k \geq 1$, where zeroes denote undercrossings and ones denote overcrossings. Notice that every alternating projection is 1-alternating. Performing appropriate Type I Reidemeister moves to an m-alternating projection of a knot results in an $(m+1)$-alternating projection [2], so a knot K is *m-alternating* provided m is the least positive integer such that K admits an m-alternating projection.

Somewhat surprisingly, every knot admits a 2-alternating projection [2]. Unlike alternating knots, 2-alternating knots may or may not admit a minimal 2-alternating projection. For example, by Theorem 7 below, $8_{19} = T_{3,4}$ admits a minimal 2-alternating projection, but no 2-alternating projection of 9_{43} is minimal because no 2-alternating projection has an odd number of crossings.

For $m \geq 2$, the *m-alternating excess of K* is

$$\chi_m(K) = \min\{c(P) - c(K) \mid P \text{ is a } m\text{-alternating projection of } K\}. \quad (3)$$

Note that any m-alternating projection of a knot K has a projection word of the form $(0^m 1^m)^k$, and hence has mk crossings. Thus, if K admits a minimal m-alternating projection, that is, if $\chi_m(K) = 0$, then m divides $c(K)$.

Torus knots provide examples of knots with minimal m-alternating projections for every $m \geq 1$, since for every $p \geq 2$, $\chi_{p-1}(T_{p,p+1}) = 0$ [2]. In fact, more can be said.

Theorem 7. *For any torus knot $T_{p,q}$ with $p < q$, we have $\chi_{p-1}(T_{p,q}) = 0$ and $\chi_{q-1}(T_{p,q}) = q - p$.*

Proof. To prove the first equality, note that one of the standard projections of $T_{p,q}$ is $(p-1)$-alternating and has $(p-1)q$ crossings. But since $p < q$, we have $(p-1)q < (q-1)p$, and so $\chi_{p-1}(T_{p,q}) = 0$.

Now consider the $(q-1)$-alternating standard projection of $T_{p,q}$. This projection has $p(q-1)$ crossings, and since $p < q$, it does not exhibit the minimal number $(p-1)q$ of crossings for $T_{p,q}$. The $(q-1)$-alternating excess for this particular projection is $p(q-1)-(p-1)q = q-p$, and therefore $\chi_{q-1}(T_{p,q}) \leq q-p$. But any $(q-1)$-alternating projection of $T_{p,q}$ with fewer than $p(q-1)$ crossings can have at most $(p-1)(q-1)$ crossings. Since $(p-1)(q-1) < (p-1)q = c(T_{p,q})$ [5], we must have $\chi_{q-1}(T_{p,q}) = q-p$. $\square$

One of the two standard braid representations of torus knot $T_{p,q}$ has braid word $(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$. For an appropriate choice of basepoint and orientation, this closed braid projection has projection word $(1^{p-1}0^{p-1})^q$. The standard closed braid representation of a spiral knot $S(n,k,\epsilon)$ has a projection word with a similar structure. If $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$, where each $\epsilon_i = \pm 1$, define the bit string $b_\epsilon = b_1 b_2 \cdots b_{n-1}$ as

$$b_i = \begin{cases} 1 & \text{if } \epsilon_i = 1 \\ 0 & \text{if } \epsilon_i = -1. \end{cases} \tag{4}$$

Let b_ϵ^R denote the reverse of b_ϵ, that is, $b_\epsilon^R = b_{n-1}b_{n-2}\cdots b_1$, and let $\bar{b}_\epsilon$ denote the 1's complement of b_ϵ. In this notation, spiral knot $S(5,3,(1,1,-1,-1))$ has $b_\epsilon = 1100$, and admits a projection word $(b_\epsilon \bar{b}_\epsilon^R)^3 = ((1100)(\overline{1100})^R)^3 = (1100)^6$; see Figure 4.

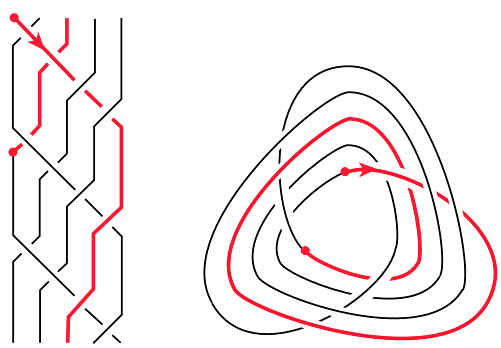

FIGURE 4. Braid representation and spiral projection of $S(5,3,(1,1,-1,-1))$ with partial projection word $b_\epsilon \bar{b}_\epsilon^R$ highlighted.

In general, traveling downwards from the basepoint at the top left of the first strand of the standard braid projection of $S(n,k,\epsilon)$ we must pass through n copies of $S(n,1,\epsilon)$ to arrive back at the first strand; this takes us through crossings $2(n-1)$ times and gives us the partial projection word $b_\epsilon \bar{b}_\epsilon^R$. Since this projection of $S(n,k,\epsilon)$ has $k(n-1)$ crossings, we must repeat the process above a total of k times to traverse each crossing twice and return to the basepoint. Therefore, the standard closed braid projection of a spiral knot $S(n,k,\epsilon)$ admits a projection word of the form $(b_\epsilon \bar{b}_\epsilon^R)^k$. We will call this the *standard projection word* for $S(n,k,\epsilon)$.

Theorem 8. *The standard closed braid projection of a spiral knot $S(n,k,\epsilon)$ is m-alternating if and only if ϵ is either of the form $1^m(-1)^m 1^m \cdots (\pm 1)^m$ or of the form $(-1)^m 1^m (-1)^m \cdots (\pm 1)^m$.*

Proof. The conditions on ϵ require that it be of one of the following possible forms: $(1^m(-1)^m)^r$ with $r \geq 1$, $((-1)^m 1^m)^r$ with $r \geq 1$, $(1^m(-1)^m)^r 1^m$ with $r \geq 0$, or $((-1)^m 1^m)^r(-1)^m$ with $r \geq 0$. These four forms of ϵ lead to, respectively, the following forms for the standard projection word $(b_\epsilon \bar{b}_\epsilon^R)^k$: $(1^m 0^m)^{2r}$, $(0^m 1^m)^{2r}$, $(1^m 0^m)^{2r+1}$, and $(0^m 1^m)^{2r+1}$, which are clearly m-alternating.

Conversely, if the standard projection of $S(n,k,\epsilon)$ is m-alternating, then it must admit a projection word of the form $(1^m 0^m)^s$. It suffices to show that the standard projection word starts with 1^m or 0^m. Suppose it begins with $1^l 0$ or $0^l 1$ with $l < m$. Then since the standard projection word has form $(b_\epsilon \bar{b}_\epsilon^R)^k$ it must terminate with either 10^l or 01^l, respectively; but a cyclic permutation of the standard projection word would then contain either $10^l 1^l 0$ or $01^l 0^l 1$, which cannot happen in an m-alternating projection. $\square$

Theorem 7 allowed us to exactly determine the m-alternating excess of torus knots for certain values of m, since the crossing numbers of torus knots are known. For spiral knots $S(n,k,\epsilon)$ where k is a prime power we have only a bound on the crossing number, which gives us a bound on the m-alternating excess of certain spiral knots.

Corollary 2. *If $k = p^r$ for some prime p and ϵ has one of the forms given in Theorem 8, then $\chi_m(S(n,k,\epsilon)) < n-1$.*

Proof. For k a prime power and ϵ as in Theorem 8, by Corollary 1 we have $\chi_m(S(n,k,\epsilon)) = (n-1)k - c(S(n,k,\epsilon)) < (n-1)k-(n-1)(k-1) = n-1$. $\square$

Acknowledgments

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References

[1] C. Adams, *The Knot Book*, American Mathematical Society, Providence, RI, 2000.
[2] P. Hackney, L. Van Wyk, N. Walters, *k-Alternating knots*, Topology and its Applications, **150** (2005), 125–131.
[3] C. Livingston, *Knot Theory*, The Mathematical Association of America, Washington, DC, 1993.
[4] K. Murasugi, *On periodic knots*, Commentarii Mathematici Helvetici, **46** (1971), 162–174.

[5] K. Murasugi, *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326** (1991), 237–260.

[6] D. Rolfsen, *Knots and Links*, Publish or Perish, Houston, 1990.

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