UNCOUNTABLY MANY MUTUALLY DISJOINT, SIMPLY CONNECTED, CONTRACTIBLE AND FRECHET DIFFERENTIABLE SUBSETS OF THE SPHERE IN ℓ^2, EACH OF WHICH IS DENSE IN THE SPHERE

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Abstract. Each sphere in ℓ^2 contains uncountably many mutually disjoint, simply connected, Frechet differentiable and contractible subsets, each of which is dense in the sphere.

1. Introduction

This paper is a sequel to the author's work in [1]. Accordingly, we shall summarize the parts of the prior paper which are satisfactory for our purposes.

Definition 1. *For $c \in \mathbb{R}$, define X_c to be the set of all real-valued sequences $x = (x_i) \in \ell^1$ such that $\sum x_i = c$.*

Definition 2. *The sphere with center x and radius r is*

$$S_{x,r} \stackrel{def}{=} \left\{ y \in \ell^2 \mid \|x - y\| = r \right\}.$$

In [1] it was shown that if $c \in \mathbb{R}$, X_c is dense in ℓ^2 and is an affine subspace of ℓ^2. Thus, $\{X_c\}_{c \in \mathbb{R}}$ is a collection of uncountably many mutually disjoint affine subsets of ℓ^2, each of which is dense in ℓ^2.

Furthermore, if $c \in \mathbb{R}$ and $S = S_{x,r}$ for an $x \in \ell^2$ and $r > 0$ is a sphere in ℓ^2, X_c is dense in S. With these preliminaries, it was shown that any sphere S in ℓ^2 contains uncountably many mutually disjoint path-connected subsets, each of which is dense in S.

In this paper, we shall not use the constructions provided in [1]. The strengthened results provided here are consequences of constructing subsets of $X_c \bigcap S$ which are more amenable to analysis than the corresponding constructions in [1].

[3] is a contemporary source of information about Hilbert spaces.

We should emphasize what we are not trying to accomplish in this and subsequent sections. We are not attempting to show that $X_c \bigcap S$ is simply connected and contractible, while being dense in S. Rather, we will show that there is a subset C_c, to be defined below, of $X_c \bigcap S$ which is simply connected and contractible, while being dense in S. As $C_c \subset X_c \bigcap S \subset X_c$, it follows that if $c \neq d$, then C_c and C_d are disjoint as they are subsets of the disjoint sets X_c and X_d, respectively.

2. Path Connectedness of C_c for $c \in \mathbb{R}$

Definition 3.

$$\pi(p_0, p_1, q) \overset{def.}{=} \textit{the plane containing } p_0, p_1 \textit{ and } q \textit{ such that } p_0, p_1 \in X_c \cap S, \\ q \in X_c, q \notin \overleftrightarrow{p_0p_1}.$$

As p_0, p_1, and q are points of X_c and X_c is affine, $\pi(p_0, p_1, q) \subset X_c$.

Definition 4.

$$\{C_c\}_{c\in\mathbb{R}} \overset{def.}{=} \{S \cap \pi(p_0, p_1, q) \mid p_0, p_1 \in X_c \cap S \\ q \in X_c, q \notin \overleftrightarrow{p_0p_1}\}.$$

Let the sphere S in ℓ^2 have center $x = (x_1, x_2, x_3, \ldots)$ and radius r. Let $p_0, p_1 \in X_c \cap S$ and $q \in X_c$ such that q is not on the line containing p_0 and p_1. Let $\pi(p_0, p_1, q)$ denote the plane containing p_0, p_1, and q. We may create a new orthonormal basis $\{e'_1, e'_2, e'_3, \ldots\}$ by letting e'_1 be $\frac{p_1 - p_0}{\|p_1 - p_0\|}$, letting e'_2 be a normalized vector in $\pi(p_0, p_1, q)$ which is perpendicular to e'_1 at p_0, and for $i \geq 3$, constructing e'_i by the Gram-Schmidt Orthonormalization process.

Note that $p_0 = (0, 0, \ldots)$, $x = (x'_1, x'_2, x'_3, \ldots)$, and a point y is in $\pi(p_0, p_1, q)$ if and only if for $i \geq 3$, $y'_i = 0$. Let $H = (x'_1, x'_2, 0, 0, \ldots)$. Define u, v, and w by:

$$\begin{aligned} u &= x - H = (0, 0, x'_3, x'_4, \ldots), \\ v &= p_0 - H = (-x'_1, -x'_2, 0, 0, \ldots), \quad \text{and} \\ w &= p_1 - H = (p'_{1_1} - x'_1, p'_{1_2} - x'_2, 0, 0, \ldots). \end{aligned}$$

Thus, $\langle u, w\rangle = \langle u, v\rangle = 0$ and $H = (x'_1, x'_2, 0, 0, \ldots)$ is the point of $\pi(p_0, p_1, q)$ nearest to x. $\|x - H\|^2 = \sum\limits_{i=3}^{\infty} (x'_i)^2 \leq r^2$, as norms are independent of the choice of orthonormal bases. Thus, H is interior to S. We now establish a lemma.

Lemma 1. *$\pi(p_0, p_1, q) \cap S$ is a circle in $\pi(p_0, p_1, q)$ with center H.*

Proof. The sphere S is the set of points $s = \{s'_1, s'_2, s'_3, \ldots\}$ such that $\sum\limits_{i=1}^{\infty} (s'_i - x'_i)^2 = r^2$. The plane $\pi(p_0, p_1, q)$ is the set of points

$s = \{s'_1, s'_2, s'_3, \ldots\}$ such that if $i \geq 3$, $s'_i = 0$. Thus, $\pi(p_0, p_1, q) \cap S$ is the set of points $s = \{s'_1, s'_2, s'_3, \ldots\}$ such that

$$\begin{aligned}\sum_{i=1}^{\infty}(s'_i - x'_i)^2 &= \sum_{i=1}^{2}(s'_i - x'_i)^2 + \sum_{i=3}^{\infty}(s'_i - x'_i)^2 \\ &= \sum_{i=1}^{2}(s'_i - x'_i)^2 + \sum_{i=3}^{\infty}(x'_i)^2 = r^2.\end{aligned}$$

The last equation above is the equation of the circle in $\pi(p_0, p_1, q)$ with center H and radius $\sqrt{r^2 - \sum_{i=3}^{\infty}(x'_i)^2}$. □

We shall refer to that circle as C. By construction p_0 and p_1 are points of the circle. Moreover $C \subset X_c$ as $C \subset \pi(p_0, p_1, q)$. Thus, $C \cap S \subset X_c \cap S$. As C is a circle, there are two arcs, which are necessarily path connected, in C connecting p_0 and p_1. Given any two points α and β of S and $\epsilon > 0$, $p_0 \in X_c \cap S$ and $p_1 \in X_c \cap S$ may be chosen such that p_0 is within ϵ of α and p_1 is within ϵ of β. That is, there is a path-connected $C \in \{C_c\}_{c\in\mathbb{R}}$ within ϵ each point of S, showing that $\{C_c\}_{c\in\mathbb{R}}$ is a collection of mutually disjoint path-connected dense subsets of S. Thus, we have the following lemma.

Lemma 2. $\{C_c\}_{c\in\mathbb{R}}$ *is a collection of uncountably many mutually disjoint path-connected subsets of S, each of which is dense in S.*

3. Simple Connectedness of C_c for $c \in \mathbb{R}$

The set-up for our work of simple connectedness is illustrated by Figure 1. To prove Simple Connectedness for the members of C_c, we shall show that any arc between p_0 and p_1 produced in the manner indicated in the prior section can be continuously transformed into any other arc between p_0 and p_1 which was also produced in that manner. Let q_0 be the point which determined the plane used to construct one such circle and let q_1 be the point which determined the plane used to construct the other circle. Adopting the convention that $a * b * c$ means that a, b, and c are collinear and b is between a and c, let q_2 such that $H * \frac{(p_0+p_1)}{2} * q_2$ and $q_2 \notin \overleftrightarrow{q_0q_1}$. As H and $\frac{(p_0+p_1)}{2}$ are points of X_c, $q_2 \in X_c$. Let $q(t)$ be defined as follows.

$$q(t) \stackrel{\text{def}}{=} \begin{cases} q_0 + 2t\,(q_2 - q_0) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ q_2 + (2t-1)\,(q_1 - q_2) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

Note that this path was chosen to avoid the possibility of there being a number t in $(0,1)$ such that $q(t)$ is a point of the line between p_0 and p_1 and of $\overleftrightarrow{q_0q_1}$. The former restriction assures that $q(t)$ may produce a plane

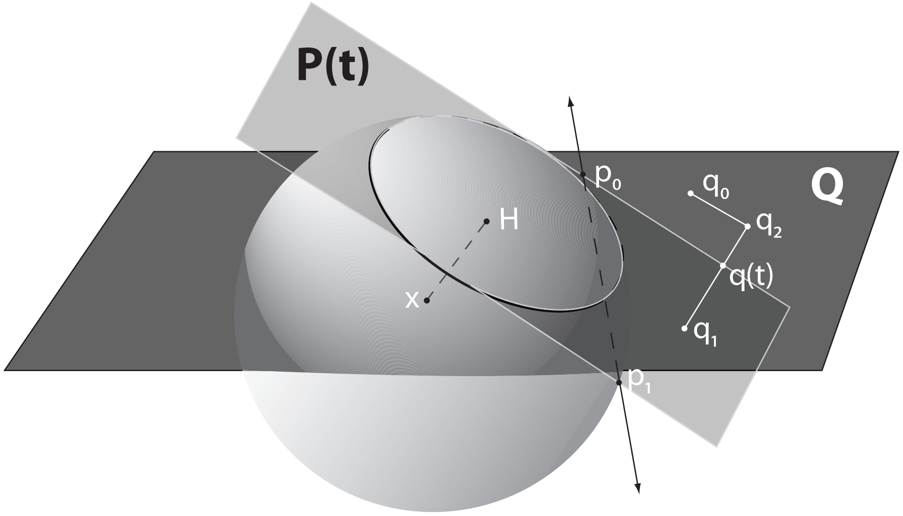

FIGURE 1. The set-up for the section on simple connectedness.

$\pi(p_0, p_1, q(t))$ and the latter restriction assures that the points $q(t)$ lie in a plane. By convexity, $q(t) \in X_c$ for all t.

Let Q denote the plane determined by $\overleftrightarrow{q_0q_2}$ and $\overleftrightarrow{q_2q_1}$. By construction, Q intersects $\overleftrightarrow{p_0p_1}$, but $\overleftrightarrow{p_0p_1}$ is not contained in Q. Thus, we may consider the following arguments in three dimensional space. We shall use $\overleftrightarrow{p_0p_1}$ as a hinge, about which planes may be produced from the three non-collinear points p_0, p_1, and $q(t)$. Let $P(t)$ denote the plane determined by p_0, p_1, and $q(t)$.

As the points of a circle C are developed continuously from its center H, H is developed from a plane by using continuous vectors and norms, planes $P(t)$ are developed continuously by pivoting points $q(t)$ around the line containing p_0 and p_1, and the points $q(t)$ are developed continuously from q_0 and q_1. It follows that the circles C are the image of a continuous function $C(t)$ of $[0, 1]$ such that $C(0)$ is the first circle and $C(1)$ the second circle. Thus, we have the following lemma.

Lemma 3. *$\{C_c\}_{c\in\mathbb{R}}$ is a collection of uncountably many mutually disjoint simply-connected subsets of S, each of which is dense in S.*

4. CONTRACTIBILITY OF C_c FOR $c \in \mathbb{R}$

The set-up for our work on contractibility is illustrated by Figure 2.

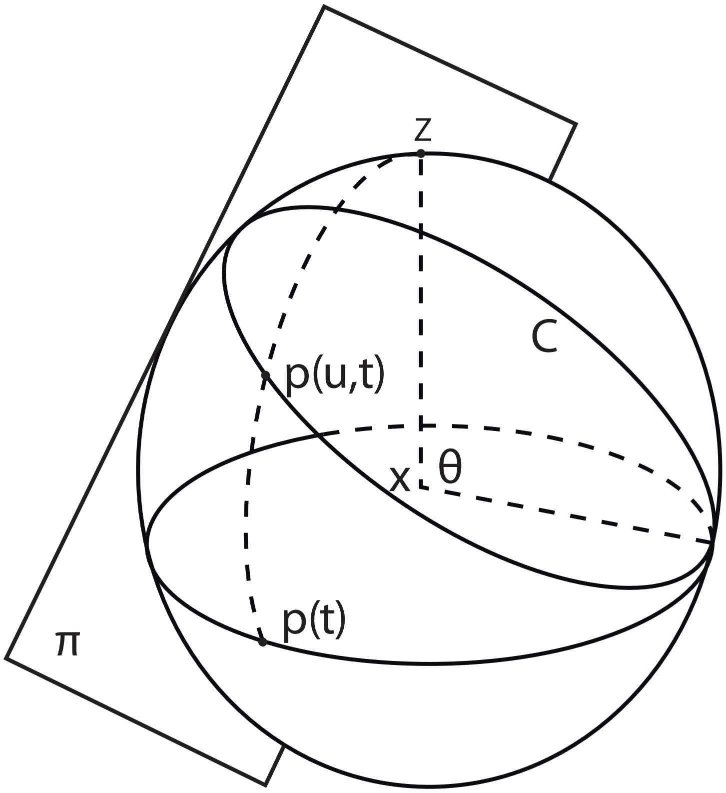

FIGURE 2. The set-up for the section on contractibility.

We will refer to a prior result holding that if $p_0, p_1 \in X_c \cap S$, then there is a point $x = (x'_1, x'_2)$ in a plane $P \subset X_c$ such that the intersection of P with S is a circle C with center $x = (x'_1, x'_2)$ containing p_0 and p_1.

Let z be any point of $X_c \cap S$, which is not a point of the circle C. We will find the angles between the rays connecting the center of the circle C with the points of the circle C and the fixed ray connecting the center x to z. As the function which denotes the angles is continuous on a closed interval, it assumes a maximum value. Then we will use that maximum angle to aid contraction. The true position of the contraction will be closer to z than the contraction under the generous estimate of the angle. That is, we will use a process similar to the Sandwich (or Squeeze) Theorem.

Part I: Let C be the circle in $X_c \cap S$ which contains two paths connecting p_0 and p_1. Parameterize the points p of C, by $p(u)$, u in $[0, 1]$. Let $\theta(u) =$

$\arccos\left(\frac{<p(u)-x,z-x>}{\|p(u)-x\|\|z-x\|}\right)$. As the denominator is positive and constant, $\theta(u)$ has a max on $[0,1]$. Let $\theta \stackrel{def}{=} \max\{\theta(u)\}, u \in [0,1]$.

Part II: For u in $[0,1]$, $\theta \geq \theta(u) > 0$. Thus, if t in $[0,1)$, $(1-t)\theta \geq (1-t)\theta(u) > 0$. Let p be a point of C. The plane Π containing p, x, and z is a subset of X_c as all three points are points of X_c and X_c is affine. An argument in the section above concerning path connectedness assures that $\Pi \cap S$ is a circle. Thus, the point set connecting p with z in Π is an arc of a circle and also a subset of $X_c \cap S$. As the curve is an arc of a circle, any ray whose initial point is interior to S intersects S only once. Thus, there is a unique point $p(t)$ on the curve such that the angle between the vectors $p(t)-x$ and $z-x$ is $(1-t)\theta$.

In like manner, there is a unique point $p(u,t)$ on the curve such that the angle between the vectors $p(u,t)-x$ and $z-x$ is $(1-t)\theta(u)$. It follows that $p(u,t)$ is between $p(t)$ and z on the arc. Thus, contracting $p(t)$ to z will contract $p(u,t)$ to z.

Part III: S has radius r. Any circle on S has radius $\leq r$. Thus, the arc length $L(t)$ on S between $p(t)$ and z is less than $(1-t)\theta r$. Or, $L(t) \leq (1-t)\theta r$. Then $\lim_{t\to 1^-} L(t) = \lim_{t\to 1^-}(1-t)\theta r = 0$. As $\|p(t)-z\| \leq$ the arc length between $p(t)$ and z, it follows $\lim_{t\to 1^-}\|p(t)-z\| = 0$. Then the remark near the end of Part II proves contractibility. That is, C_c is contractible. Thus we have the following lemma.

Lemma 4. *$\{C_c\}_{c\in\mathbb{R}}$ is a collection of uncountably many mutually disjoint contractible subsets of S, each of which is dense in S.*

We will now show that if $C \in C_c$, C is Frechet differentiable.

Lemma 5. *For $c \in \mathbb{R}$, each path in C_c is Frechet Differentiable.*

Proof. Gamelin and Greene [2, p. 47] offers a definition of Frechet Differentiability in Banach spaces and hence in ℓ^2. For convenience, we shall refer to the expression $x - x_0$ on that page as h. By construction, each path C in C_c is a circle. Utilizing a process carried out earlier, we may construct an orthonormal basis with origin at the center of C and whose first two members are contained in the plane containing C. In that system $x \in C$ implies $\|x\| = r$ or $G(x) = \langle x,x\rangle = r^2$. ℓ^2 is the open subset of ℓ^2 required by the definition. Thus,

$$\lim_{h\to 0}\frac{G(x)-G(x-h)-T(h)}{\|h\|} \tag{1}$$

$$= \lim_{h\to 0}\frac{\langle x,x\rangle - [\langle x,x\rangle - 2\langle x,h\rangle + \langle h,h\rangle] - T(h)}{\|h\|}.$$

Setting $T(h)$ to $2\langle x,h\rangle$, expression (1) becomes

$$\lim_{h\to 0}\frac{-\langle h,h\rangle}{\|h\|}=\lim_{h\to 0}\frac{-\|h\|^2}{\|h\|}=0.$$

□

Thus, C is a Frechet differentiable path and its derivative is $2\langle x,h\rangle$. Finally, we have the following theorem.

Theorem 1. $\{C_c\}_{c\in\mathbb{R}}$ *is a collection of uncountably many mutually disjoint, simply connected, Frechet differentiable and contractible subsets of* S*, each of which is dense in* S*.*

References

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AMS Classification Numbers: 46B20, 46C05

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