STRUCTURING UNDERGRADUATE RESEARCH IN ABSTRACT ALGEBRA

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Abstract. We discuss some general methods for structuring undergraduate research projects. As an example, we follow a project which occurred between the two authors who at one time had the student-advisor relationship. We discuss realistic goals of undergraduate research, reflect on the methods and outcomes of our project, and suggest ideas for future work with undergraduates.

1. Introduction. Directing undergraduate research can be quite challenging for many working college professors. There is the common difficulty of proposing problems suitable for an undergraduate student. On one hand, you do not want the problem to be too difficult. On the other hand, the problem should be just slightly out-of-reach of the student's current knowledge. This will force the student to spend time learning new material while working in the direction of a solution. These factors make the project feel like research to the undergraduate student, the primary purpose of undergraduate research in our opinion, even though the results may not be worthy of a highly regarded research journal.

As the authors of this article, we are in a rather unique position to comment on the experience of undergraduate research. While both of us hold a Ph.D. in mathematics and teach at liberal arts colleges, this was not always the case. We once had the student-advisor relationship. In 1995, the first author completed an undergraduate research project in abstract algebra under the advisement of the second author. We would like to take this opportunity to reflect on the project, share the results, and offer some suggestions for future projects.

The topic of our project focused on the structure of rings, a topic that is not normally discussed during an introductory abstract algebra course at the undergraduate level. The approach was to take a rather basic concept which arises in any undergraduate curriculum and to get the student to experience a generalization of that concept. In our case, the basic concept was that of the transpose of a matrix: an involution (*) of a ring R is an anti-automorphism of order two; that is, (*) is a group automorphism of order two of the additive group $(R,+)$ satisfying $(r_1r_2)^* = r_2^*r_1^*$ for all $r_i \in R$. The transpose on matrices can be easily seen as an example of an involution on the ring of matrices.

We will now share our own experiences during this undergraduate research project. It is our sincere hope that this will give a guideline for a successful supervision of undergraduate research particularly for those who have just started their career in teaching at liberal arts colleges such as ours.

2. Getting Started: Definitions and Basic Results. The first step in our project was to introduce the generalization including the setting in which the generalization resides. In our case, matrices were generalized to a general non-commutative ring and the following definition was introduced by the student.

Definition 2.1. A mapping $^*\colon R \to R$ of a ring R onto itself is called an involution if for all $a, b \in R$,

1. $(a^*)^* = a$
2. $(a+b)^* = a^* + b^*$
3. $(ab)^* = b^* a^*$.

We will denote the ring R with involution (*) by R^* hereafter. Also, for a subset L of R^*, the set $\{r^* : r \in L\}$ will be denoted by L^*. An ideal I of R^* is called a (*)-ideal if $I^* \subseteq I$, and a (*)-ideal P is called (*)-prime provided that $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$ for (*)-ideals I and J. Also, a proper (*)-ideal M is said to be (*)-maximal if there does not exist a (*)-ideal N such that $M \not\subseteq N$.

After a reading course in abstract algebra, Keith (the student) was able to find some basic properties of rings with involution as listed below.

1. If I is a (*)-ideal, then $I^* = I$.
2. A power and an intersection of (*)-ideals is a (*)-ideal.
3. $1 = 1^*$
4. If I is an ideal of R, then so is I^*, and hence II^*, $I + I^*$, and $I \cap I^*$ are all (*)-ideals.

During the reading course, these properties were given to Keith as open-ended questions. For example, the last statement was initially asked as "If I is an ideal of R, what can you say about II^*, $I + I^*$, and $I \cap I^*$?" This, we believe, is a great way for a student to start digesting the new definitions. In addition, Keith quickly realized that there is a fundamental difference between doing homework problems and writing a paper. For example, you need to determine what to prove before trying to prove it. These properties were essential tools to prove many of the propositions and theorems that appear in the project.

3. I Need an Example! For a student, working in this degree of abstraction can be both enlightening and extremely frustrating. The time will come where one must see examples. Through some discussions, Hisa (the advisor) was eventually able to convince Keith that examples exist. Moreover, different kinds of examples were discussed.

The motivating example of an involution is the transpose on matrices which is inextricably bound in with the structure of matrices.

Keith: "Are there any other examples of a ring with involution?"

Hisa: "Think about the identity map defined on a ring."

For an adviser, working with a student on an undergraduate research project can be both enlightening and extremely frustrating as well. How much should we give away and how much should we keep to ourselves? One thing that worked out well in our case was not to disprove the student's wrong conclusion, but to let the student naturally realize it.

Keith: "I see, it looks like we can define an involution on any ring."

Hisa: "I am not sure if I would go that far ... what if a ring is non-commutative? Let's see ..."

Hisa then suggested that Keith try to strengthen his conclusion by first considering defining an involution on some non-commutative ring of small order, hoping that Keith would eventually come up with a counter example. Keith struggled for a few days trying to prove his claim but then was able to verify that no matter how the (*) operation is defined on the ring shown below, the last involution property will not be satisfied. Therefore, he found a ring in which an involution cannot be defined.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Notice, however, that no identity is present in the ring. Is it then possible to construct a ring with identity in which no involution can be defined? Soon Keith recalled the procedure which constructs a ring with identity from a ring without identity, a procedure mentioned in his abstract algebra course. With the previously defined ring as a starter, Keith considered a set of ordered pairs whose first component comes from the ring previously defined and whose second component comes from the two-element ring $\mathbb{Z}_2$. Addition is performed by coordinates, and multiplication is defined by $(x_1, y_1)(x_2, y_2) = (x_1x_2 + y_1x_2 + x_1y_2, y_1y_2)$ for all $x_1, x_2 \in R$ and $y_1, y_2 \in \mathbb{Z}_2$. This set forms a ring with identity (0,1). Keith was then able to show that no involution can be defined on this ring.

Just as Keith had shown that there exist rings in which no involution can be defined, Hisa suggested the construction of a ring with involution from any given ring, with or without identity. For such a task, take any ring R and redefine its multiplication as follows: for any $r_1, r_2 \in R$, $r_1 \star r_2 = r_2 \cdot r_1$. The new operation $\star$ simply reverses the operands in the original operation. With the same additive structure, it is obvious that R with this new operation forms a ring. We will refer to a ring R with this redefined multiplication as R^{op} (the *opposite ring*). For any ring R, it is known and easy to see that the new ring $R \oplus R^{op}$, with pairwise addition and multiplication, can always have an involution.

We are finally ready for Keith to try his own hand at research. We cannot emphasize enough that the work up to this point takes a good deal of time. Students should plan to spend half of the semester or more getting to this point. The student should learn to determine some basic results, form examples, and formulate his/her own conjectures. Only after this amount of exposure to the new concept can an undergraduate student begin thinking about his or her own theorems.

4. Main Results. A natural starting place is the concept of a simple ring. Recall that a ring is called *simple* if it does not contain any non-zero proper ideals, and so a ring with involution having no non-zero (*)-ideals will be called a *(*)-simple* ring. Moreover, rings with involution (*) in which every (*)-ideal is (*)-prime will be referred to as fully (*)-prime rings. Rings with involution (*) in which every non-zero (*)-ideal is (*)-maximal will be called fully (*)-maximal rings. Keith's first result was to generalize the following well-known characterization of simple rings and (*)-simple rings. This is, of course, a perfect place to start since it builds on the student's prior knowledge.

Proposition 4.1. The ring R is a simple ring if and only if $R \oplus R^{op}$ is (*)-simple.

Coming up with the statements of new theorems is very likely out-of-reach for most undergraduates. This is where the advisor should step in and suggest a possible result.

Hisa: "Say, I wonder how R, being fully prime, will affect $R \oplus R^{op}$? Maybe you should think about it?"

Keith: "Oh, ok Dr. T. I'll work on that tonight while I'm watching Seinfeld."

... *a few days later* ...

Keith: "Sorry to let you down, Dr. T, but I couldn't figure anything out."

Hisa: "Well, at least you thought about it. How about this: maybe if R is fully prime, then $R \oplus R^{op}$ will be fully (*)-prime. See if you can prove it tonight while you watch Melrose Place. If not, maybe you can find a counter example to show that it's false."

And two weeks later, Keith arrives with the following proposition.

Proposition 4.2. The ring R is fully prime if and only if $R \oplus R^{op}$ is fully (*)-prime.

Since the purpose of this article is not to show you the proofs to some theorems in ring theory, we are omitting the proofs. The student might not (in fact, probably will not) write the proof perfectly the first time. As long as the undergraduate student is thinking and trying to write some arguments, the advisor should be happy. Certainly, we rarely (if ever) had the experience of Keith writing an entire proof with no help from Hisa.

Keith considered fully (*)-maximal rings at the end. From our independent reading course, Keith knew that every maximal ideal is prime. An almost identical proof of the statement yields the following (*) analog.

Proposition 4.3. Let R be a ring with involution (*). Any (*)-maximal ideal M of R is a (*)-prime ideal.

So, notice that the results are very natural generalizations of work already known to the student. Hisa has simply taken known theorems and asked Keith to apply them to a new structure, a ring with involution. This process can work very well if the advisor can conjure up an appropriate object to generalize. Our final discussion was about the structure of rings with involution, all of whose (*)-ideals are (*)-maximal. Such a ring has a quite limited structure as the following theorem shows.

Theorem 4.4. A fully (*)-maximal ring R has at most two proper (*)-ideals.

This theorem eventually led Keith to a final nice result, Theorem 4.5. We include Keith's proof for good measure. Notice how the student has pulled together the idea of rings with involution (the theme of the project) along with some results from the reading courses (Chinese Remainder Theorem, and the Fundamental Theorem of Ring Homomorphisms). This, we believe, is a wonderful way to end the experience.

Theorem 4.5. Let R be a fully (*)-maximal ring. Then R is either:

1. a (*)-simple ring,
2. a ring with exactly one non-zero proper (*)-ideal, or
3. the direct sum of two (*)-simple rings.

Proof. We have already shown that a fully (*)-maximal ring has at most two (*)-ideals. All that is left to show is the claim that a ring with exactly two non-zero (*)-maximal ideals is the direct sum of two (*)-simple rings. Let M_1 and M_2 be distinct non-zero (*)-maximal ideals. Since $M_1 + M_2$ is a (*)-ideal, we must have $M_1 + M_2 = R$. By the Chinese Remainder Theorem, the map $R \to R/M_1 \oplus R/M_2$ is onto. But the kernel of this map is $M_1 \cap M_2$ which is zero, since $M_1 \cap M_2$ is a (*)-ideal of the ring. Hence, by the Fundamental Theorem of Ring Homomorphisms, $R \simeq R/M_1 \oplus R/M_2$.

5. Wrapping Things Up. A good undergraduate project should have the student believing that there's still more work to be done. The student should be able to see ahead slightly and understand how to start asking his or her own questions about what to study next. Of course, the advisor might coerce this, but students should certainly not feel that the research is done. The results obtained here naturally lead us to a few questions.

- Is a (*)-simple ring a simple ring?
- Is a (*)-maximal ideal within a fully (*)-maximal ring maximal?

After the project and final paper were completed, Keith was able to find some answers to these questions. To answer the first, consider the ring $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ with exchange involution. This ring has two distinct non-zero ideals, namely $\mathbb{Z}_3 \oplus \{0\}$ and $\{0\} \oplus \mathbb{Z}_3$, neither of which is a (*)-ideal. So, (*)-simple does not imply simple.

For the answer to the second question, consider the lattice of ideals for the ring $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ with typical exchange involution. Notice that the ring has just one non-zero (*)-ideal, namely $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(a, b) : a, b \in \{0, 2\}\}$. But this ideal is not maximal, since it is contained within $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. Hence, (*)-maximal does not imply maximal. The one case from our proposition which is left for further study is the structure of a ring with involution having just one non-zero (*)-ideal.

6. Creating New Projects. At the time when Keith was working on this project, Hisa was working on his own research on a related subject. He, therefore, had several ideas and minor results which best served as tools for Keith's project rather than as a weak publication of his own. We think most new Ph.D.s have similar ideas and minor results that may be useful for conducting undergraduate research. This might require some thought, but this is to be expected. Our experience has been that you cannot find an undergraduate research problem easily.

Our approach in this project was to *generalize*. Hisa took a basic concept in abstract algebra and asked Keith to generalize it (slightly) and sort of "reprove" the known theorems. You should also think about moving in the other direction. In our own research, we often look for general statements like "For any field F, ..." But perhaps a more restrictive statement will be accessible to an undergraduate. For instance, what if F were restricted to the field of real numbers, or only rationals, or even a finite field of prime order? Perhaps this specification will lead to a more accessible problem.

We cannot overemphasize that the value and motivation of undergraduate research lies more heavily on the "research experience" than any "new" results. A student who is interested in working on a project is likely a good and motivated student. Eventually, he or she will find his or her own solid area to pursue under the guidance of a graduate adviser. The idea is to project the guidance that you had or wished to have had during your graduate research days to the undergraduate student in the time prior to his or her entry into the graduate research world.

We shall now list a few other projects within the area of abstract algebra for the reference of interested readers. We believe this approach (generalizing a concept) works successfully in many other areas of mathematics. The topics listed below are fairly well-known, but students might need to "research" a bit to come across the right articles to reference. As it was in our case, we believe these topics will strengthen a student's solid understanding of basic linear algebra material. They also let the student rethink about the oft-neglected relation between abstract and linear algebra courses. It is very possible that a student may even find a small new result although we do not think that is a priority in undergraduate research.

1. Consider the ideal structure of the ring of linear transformations of a vector space over a field as a generalization of the ring of matrices over a field.
2. Generalize involution further to a map of anti-homomorphisms of order 2 and investigate the structure of rings where the map can be defined.
3. A combination of (1) and (2): study the fundamental property known as "Invariant Base Number." A field has this property (every base of a vector space has the same cardinality). The ring in (1) gives an example of a ring without the property, while the ring in (2) with an additional condition guarantees an invariant base number.

It is our hope that we have given at least some suggestion on how to proceed with your own undergraduate research projects in abstract algebra. Undergraduate research has become quite the buzz word these days in institutions striving toward quality undergraduate education, but there is little training on how to proceed with such projects. To date, we have both supervised several projects and are finally starting to feel knowledgeable about the process. We hope that other teachers of undergraduate mathematics can benefit from our experience and ideas. Let us know how you make out.

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