RETURNS TO THE ORIGIN FOR RANDOM WALKS ON $\mathbb{Z}$ REVISITED

Helmut Prodinger

Chrysafi and Bradley [1] considered symmetric random walks, defined as follows: let $X_k, k = 1, 2, \dots$ be independent and identically distributed random variables with $\mathbb{P}\{X_k = 1\} = \mathbb{P}\{X_k = -1\} = \frac{1}{2}$. Then

$$S_m = \sum_{k=1}^{m} X_k \quad \text{with} \quad S_0 = 0$$

is a simple random walk starting at 0. The authors considered only walks of even length $m = 2n$ and were interested in the random variable $R = R_n$, defined to be the *number of returns to the origin* in a walk of length $2n$, i.e., the number of times $S_i = 0$ happens, for $i = 1, \dots, 2n$. They computed moments up to $\mathbb{E}[R^6]$ and asked for a closed formula for $\mathbb{E}[R^k]$ and also whether $\mathbb{E}[R^k] \sim c_k n^{k/2}$ holds.

The answers to these questions can be found in [4] as opposed to [1]. There, the *factorial moments* $\mathbb{E}[R^k]$ were computed. We state the formula only for even n:

$$\mathbb{E}[R_n^k] = k! \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{\frac{i}{2} + n}{n}.$$

Ordinary moments can be recovered from these formulae as linear combinations with *Stirling numbers of the second kind (Stirling subset numbers),* see [3].

$$\mathbb{E}[R_n^k] = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} \mathbb{E}[R_n^i].$$

To answer the asymptotics question, we use generating functions

$$\mathbb{E}[R_n^k] = 4^{-n} k! [z^n] \frac{(1 - \sqrt{1-4z})^k}{(1-4z)^{1+\frac{k}{2}}},$$

and *singularity analysis of generating functions,* as nicely described in [2]. One must expand around the dominant singularity (here $z = \frac{1}{4}$),

$$\frac{(1-\sqrt{1-4z})^k}{(1-4z)^{1+\frac{k}{2}}} \sim (1-4z)^{-1-\frac{k}{2}},$$

and use a transfer theorem

$$\mathbb{E}[R_n^k] \sim 4^{-n} k! [z^n](1-4z)^{-1-\frac{k}{2}} = k! \binom{\frac{k}{2}+n}{n} \sim \frac{k!}{(k/2)!} n^{k/2}.$$

For odd $k = 2j+1$, the factor may be rewritten as follows:

$$\frac{k!}{(k/2)!} = \frac{(2j+1)!}{(j+\frac{1}{2})!} = \frac{2^{2j+1} j!}{\sqrt{\pi}}.$$

For ordinary moments, the leading terms in the asymptotic expansion are the same.

$$\mathbb{E}[R_n^k] \sim \frac{k!}{(k/2)!} n^{k/2}.$$

Acknowledgement. This research was done while the author was with The John Knopfmacher Centre for Number Theory and Applicable Analysis at the University of the Witwatersrand, Johannesburg, South Africa.

References

1. L. A. Chrysafi and R. E. Bradley, "Returns to the Origin for Random Walks on $\mathbb{Z}$," *Missouri Journal of Mathematical Sciences,* 14 (2002), 96–106.
2. P. Flajolet and A. Odlyzko, "Singularity Analysis of Generating Functions," *SIAM Journal on Discrete Mathematics,* 3 (1990), 216–240.
3. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science,* Addison-Wesley Publishing Company, Amsterdam, 2nd edition, 1994.

4. P. Kirschenhofer and H. Prodinger, "The Higher Moments of the Number of Returns of a Simple Random Walk," *Adv. in Appl. Probab.,* 26 (1994), 561–563.

5. P. Kirschenhofer and H. Prodinger, "Return Statistics of Simple Random Walks," *J. Statist. Plann. Inference,* 54 (1996), 67–74.

Mathematics Subject Classification (2000): 60C05, 05A15

Helmut Prodinger
Mathematics Department
Stellenbosch University
7602 Stellenbosch, South Africa
email: hproding@sun.ac.za